Holographic hydrodynamics

(lecture 2)

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Hydrodynamics as an effective field theory

$\Longrightarrow$ Consider (relativistic) translationary invariant system in flat space time. We will assume that the system interacts with a thermal bath of temperature T; it might have a set of intrinsic scales m_i

$\Longrightarrow$ We would like to construct a most general effective description of the system, valid on distances ℓ and time-scales τ much much larger than the intrinsic scales

$$\frac{1}{\ell} \equiv |\vec{k}| \ll \min\{T, m_i\}\,, \qquad \frac{1}{\tau} \equiv \omega \ll \min\{T, m_i\}$$

$\Longrightarrow$ a theory providing such a description is **Relativistic Hydrodynamics**

$\Longrightarrow$ Hydrodynamics owes its existence to the presence of conserved quantities in the system — the stress energy tensor $T_{\mu\nu}$,

— the conserved currents J^μ associated with the global $U(1)$ charges

it is a theory of slow and gradual variation of these quantities:

$$\nabla_\mu T^{\mu\nu} = 0\,, \qquad \nabla_\mu J^\mu = 0$$

How do we construct such an effective description?

For a system in equilibrium, in a local rest frame:

$$T_{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}, \qquad J^\mu = \rho\ \delta^\mu_0$$

where ϵ is the energy density, P is a pressure and ρ is a global $U(1)$ density. In any other reference frame, related to above by a Lorentz transformation with a (constant) time-like 4-velocity u^μ, $u^\mu u_\mu = -1$:

$$T^{\mu\nu} = \epsilon\ u^\mu u^\nu + P\ \Delta^{\mu\nu}\,, \qquad J^\mu = \rho\ u^\mu$$

with

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$$

being a symmetric transverse tensor —

$$u_\mu \Delta^{\mu\nu} = 0$$

Suppose that a system is *slightly* off-equilibrium:

$$\epsilon = \epsilon(t, \vec{x})\,, \qquad P = P(t, \vec{x})\,, \qquad \cdots$$

Slightly means that we still have local equilibrium, and thus a familiar thermodynamic relations

$$\epsilon + P = sT + \mu\rho\,, \qquad d\epsilon = T ds + \mu d\rho$$

but the equilibrium is not yet reached globally — specifically,

$$T^{\mu\nu} = \epsilon\; u^\mu u^\nu + P\; \Delta^{\mu\nu} + \delta T^{\mu\nu}\,, \qquad J^\mu = \rho\; u^\mu + \delta J^\mu$$

where

$$\delta T^{\mu\nu} = \delta T^{\mu\nu}\{u^\mu\,, T\,, \mu\,; \nabla u^\mu\,, \nabla^2 u^\mu\,, \cdots\}\,, \qquad \delta J^\mu = \delta J^\mu\{\cdots\}$$

with $u^\mu = u^\mu(t, \vec{x})$. We use

$$T\,, \qquad \mu\,, \qquad u^\mu$$

as an independent variables — fields of the effective description

There is a freedom of choosing the local reference frame, and we can do so that the corrections $\delta T^{\mu\nu}$ and δJ^{μ} are transverse

$$u_\mu \delta T^{\mu\nu} = u_\mu \delta J^\mu = 0$$

If the fundamental variables are slowly varying in space-time we can — following the logic of the low-energy effective field theory — organize $\delta T^{\mu\nu}$ and δJ^μ into gradient expansion of independent variables (fields). The most general transverse expansion to leading order in the derivatives

$$\delta T^{\mu\nu} = -\eta \left[\Delta^{\mu\lambda}\left(\nabla_\lambda u^\nu + \nabla^\nu u_\lambda - \frac{2}{3}\delta^\nu_\lambda \nabla_\alpha u^\alpha\right)\right] - \zeta \left[\Delta^{\mu\nu}\ \nabla_\alpha u^\alpha\right] + \cdots$$

$$\delta J^\mu = \sigma_Q \left[\mu \Delta^{\mu\nu}\ \nabla_\nu \ln \frac{T}{\mu}\right] + \cdots$$

where we choose a conventional parametrization. Thus to leading order in the derivative expansion we have 3 independent terms ($\equiv$ operators) and 3 transport coefficients ($\equiv$ couplings):

$$\eta\,, \qquad \zeta\,, \qquad \sigma_Q$$

the shear viscosity, the bulk viscosity, the charge conductivity

$\cdots$ we can proceed with constructing higher order corrections $\cdots$

Kubo formula for shear viscosity and hydrodynamic modes

$\Longrightarrow$ Kubo formula for shear viscosity

Consider a deformation of the theory S_0 by a source term $J_a(x)$ for an operator $\mathcal{O}_a(x)$:

$$S = S_0 + \int_x J_a(x)\mathcal{O}_a(x)$$

Linear response theory states that expectation values of $\mathcal{O}_a$ in the deformed theory are proportional to the sources J_b

$$\langle \mathcal{O}_a(x) \rangle = -\int_y G^R_{ab}(x-y) J_b$$

there the retarded Green's function $G^R_{ab}(x-y)$ can be evaluation in undeformed theory:

$$iG^R_{ab}(x-y) = \theta(x^0 - y^0)\ \langle [\mathcal{O}_a(x), \mathcal{O}_b(y)] \rangle \Big|_{S_0}$$

$\Longrightarrow$ Consider the special case of the retarded correlation function of the stress-energy tensor. A source here is a metric perturbation $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

- Consider the metric perturbation which are traceless $h^{\mu}_{\mu} = 0$, and spatially homogeneous

$$h_{00}(x^{\mu}) = 0\,, \qquad h_{0i}(x^{\mu}) = 0\,, \qquad h_{ij}(x^{\mu}) = h_{ij}(x^0)$$

- We use hydrodynamics to compute the response of the stress-energy tensor. Note that because the source is spatially homogeneous, we must have

$$u^{\mu} = u^{\mu}(x^0)\,, \qquad \text{of course}: \qquad u^{\mu}u_{\mu} = -1$$

  Parity invariant along the spatial directions $u^i \leftrightarrow -u^i$ further implies that for the particular source,

$$u^{\mu} = (1, 0, 0, 0)$$

  i.e, the fluid remains at rest.

- Plugging in the metric to linear order and the 'rest' 4-velocity into

$$\delta T^{\mu\nu} = -\eta\left[\Delta^{\mu\lambda}\left(\nabla_\lambda u^\nu + \nabla^\nu u_\lambda - \frac{2}{3}\delta^\nu_\lambda \nabla_\alpha u^\alpha\right)\right] - \zeta\left[\Delta^{\mu\nu}\ \nabla_\alpha u^\alpha\right]$$

  we find

$$\delta T_{ij}(t) = 2\eta\Gamma^0_{ij} = -\eta\ \partial_0 h_{ij}(x^0) + \mathcal{O}(\partial_0^2)$$

  where Christoffel connections come entirely from the metric perturbation. The rest of the stress-energy components vanish.

- For $h_{ij} = e^{-i\omega x^0} H_{ij}$ (H_{ij} are constants) the linear response states:
  - for the LHS,

$$\delta T_{ij} = i\eta\ \omega\ e^{-iwx^0} H_{ij} + \mathcal{O}(\omega^2)$$

  - for the RHS,

$$\delta T_{ij} = -\int dy^0 d\vec{y} G^R_{ij,ij}(x^0 - y^0, \vec{0} - \vec{y}) e^{-i\omega y^0} H_{ij}$$

■ $\Longrightarrow$

$$i\eta\ \omega = -\int dy^0 d\vec{y} e^{i\omega(x^0-y^0)} G^R_{ij,ij}(x^0-y^0,\vec{0}-\vec{y}) + \mathcal{O}(H)$$

or shifting $y^0 \equiv y^0 - x^0$

$$i\eta\ \omega = -\int dy^0 e^{-i\omega y^0} G^R_{ij,ij}(y^0,\vec{y}) + \mathcal{O}(\omega^2) = -\hat{G}^R_{ij,ij}(\omega,\vec{0}) + \mathcal{O}(\omega^2)$$

where the $\hat{}$ denotes the Fourier component of the correlation function.

- Thus, we arrive at Kubo formula

$$\eta = -\lim_{\omega\to 0} \frac{1}{i\omega} \hat{G}^R_{ij,ij}(\omega,\vec{0}) = -\lim_{\omega\to 0} \frac{1}{\omega} \mathrm{Im}\, \hat{G}^R_{ij,ij}(\omega,\vec{0})$$

$\Longrightarrow$ Hydrodynamic modes

We would like now to study spectrum of hydrodynamics fluctuations around the equilibrium thermal state in which

$$u^\mu = (1,0,0,0)\,, \qquad T = \text{const}$$

- As an independent set of variables we will choose the three spatial components of the velocity $\delta u^1 = \delta u^x$, $\delta u^2 = \delta u^y$, $\delta u^3 = \delta u^z$, as well as δT.
- We assume that all the perturbations are of the plane-wave form

$$e^{-i\omega t + iqz}$$

- We can compute the relevant fluctuations of $T^{\mu\nu}$ to linear order in δu^i and δT:

$$\delta T^{tt} = \delta\epsilon = \left(\frac{\partial \epsilon}{\partial T}\right)\,\delta T$$

$$\delta T^{ti} = (\epsilon + P)\delta u^i$$

$$\delta T^{xz} = -\eta \partial_z \delta u^x \,, \qquad \delta T^{yz} = -\eta \partial_z \delta u^y$$

$$\delta T^{zz} = \delta P - \left(\frac{4}{3}\eta + \zeta\right) \partial_z \delta u^z = \left(\frac{\partial P}{\partial T}\right) - \left(\frac{4}{3}\eta + \zeta\right) \partial_z \delta u^z$$

- Use

$$\partial_z \delta u^j = iq \ \delta u^j \,, \qquad \partial_t \delta u^j = -i\omega \ \delta u^j$$

- Evaluating hydrodynamic EOMs

$$\partial_\mu \delta T^{\mu\nu} = 0$$

we find the decoupled system of equations:

- sound waves in plasma

$$\begin{cases} 0 = \omega \left(\frac{\partial \epsilon}{\partial T}\right) \ \delta T - q(\epsilon + P) \ \delta u^z \\ 0 = \omega(\epsilon + P) \ \delta u^z - q \left(\frac{\partial P}{\partial T}\right) \ \delta T + iq^2 \left(\frac{4}{3}\eta + \zeta\right) \ \delta u^z \end{cases}$$

- shear modes of different polarization:

$$0 = \Big(\omega(\epsilon + P) + iq^2 \eta\Big) \ \delta u^{x,y}$$

- Solving above equations we find the spectrum in the hydrodynamics (small ω, q) limit:
  - for the sound channel

$$\omega = \pm c_s q - i\Gamma q^2 + \mathcal{O}(q^3)$$

where

$$c_s^2 = \frac{\partial P}{\partial \epsilon}, \qquad \Gamma = \frac{2\eta}{3(\epsilon + P)}\left(1 + \frac{3\zeta}{4\eta}\right)$$

  - for the shear channel

$$\omega = -iq^2 \frac{\eta}{\epsilon + P} = -i\frac{q^2}{T}\,\frac{\eta}{s}$$

$\Longrightarrow$ Note: the shear modes are not propagating (they are diffusive)

Hydrodynamics of $\mathcal{N}=4$ SYM plasma

$\Longrightarrow$ We now discuss hydrodynamics of strongly coupled $\mathcal{N}=4$ plasma from the dual gravitational perspective

Kubo formula

$\Longrightarrow$ Recall that Kubo formula implies computation of the retarded correlation functions. I will outline how this is done in holography.

- Gauge theory partition function in the presence of a source J is a generating functional of the connected correlation functions of the corresponding operators:

$$Z_{gauge}[J]=\int[d\phi]exp\left(iS_{gauge}+i\int d^4xJ\mathcal{O}\right)$$

$$G^{(n)}(x_1,\cdots x_n)\equiv -i\langle T\left(\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right)\rangle=\frac{i\delta^n\ \ln Z_{gauge}[J]}{\delta J(x_1)\cdots J(x_n)}$$

- At strong coupling the gauge invariant operator $\mathcal{O}$,

$$\dim[\mathcal{O}] = \Delta$$

is dual to a classical, on-shell configuration of the corresponding gravitational field $\phi_{cl}(x;r)$, with the asymptotic of the AdS radial coordinate $r \to \infty$

$$\phi_{cl}(x;r) = r^{-\Delta} J(x) + \cdots$$

Furthermore,

$$Z_{gauge}[J] = Z_{string} \approx e^{iS_{SUGRA}[\phi_{cl}]}\Big|_{\lim_{r\to\infty} r^{\Delta}\phi_{cl}(x;r)=J(x)}$$

leading to

$$G^{(n)}(x_1, \cdots x_n) \equiv -i\langle T\left(\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\right)\rangle = -\frac{\delta^n \; S_{SUGRA}[\phi_{cl}]}{\delta J(x_1)\cdots J(x_n)}$$

$\Longrightarrow$ Remember that S_{SUGRA} should be evaluated on the equation of motion of ϕ_{cl} with the boundary condition as dictated by the source.

$\Longrightarrow$ As an example, consider the two-point correlation function of the stress-energy tensor in Minkowski vacuum.

- The role of the source for the stress energy tensor is the boundary metric. Recall,

$$T_{ij} = \frac{2}{\sqrt{-g^{(0)}}} \frac{\delta S}{\delta g^{ij(0)}}$$

- In holography, we perturb the AdS metric

$$ds_5^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{r^2}{L^2} \eta_{ij} dx^i dx^j + \frac{L^2}{r^2} dr^2$$

as

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$$

- Corresponding to the above bulk metric perturbation,the boundary metric is perturbed as

$$\delta g_{ij}^{(0)} = \lim_{r\to\infty} \frac{L^2}{r^2} h_{ij}$$

- To simplify the problem further (this ultimately restricts correlation functions of $T_{\mu\nu}$ with which indexes we can compute), we assume

$$h_{\mu\nu} = h_{\mu\nu}(t, z, r)$$

- We would need to solve classical linear equations of motion for $h_{\mu\nu}$. An above ansatz has an $O(2)$ rotational symmetry around z-direction. This symmetry implies what fluctuations or different helicity with respect to this symmetry would decouple. Here are the decoupled combinations:
  - helicity-2: $\{h_{xy}\}$
  - helicity-2: $\{h_{xx} - h_{yy}\}$
  - helicity-1: $\{h_{tx}, h_{xz}, h_{xr}\}$
  - helicity-1: $\{h_{ty}, h_{yz}, h_{yr}\}$
  - helicity-0: $\{h_{xx} + h_{yy}, h_{zz}, h_{tt}, h_{tz}, h_{tr}, h_{zr}, h_{rr}\}$

  Decoupling implies that each of the sets can be considered separately.
- We assume that only h_{xy} fluctuations are non-vanishing.

- Define ϕ as

$$\phi(t,z,r) \equiv h_x^y = \frac{L^2}{r^2} h_{xy}$$

Note, with such definition,

$$\delta g_{xy}^{(0)} = \lim_{r\to 0} \phi(t,z,r)$$

- The quadratic effective action for the fluctuations is simply the one for the minimally coupled massless bulk scalar ϕ,

$$S_{SUGRA} = \frac{N^2}{8\pi^2 L^3} \int d^5x \sqrt{-g} \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right)$$

where $g_{\mu\nu}$ is unperturbed bulk metric. In above we expressed G_5 in gauge theory parameters

$$\frac{1}{16\pi G_5} = \frac{N^2}{8\pi^2 L^3}$$

- ϕ satisfies EOM:

$$\Box\phi = 0$$

or with the ansatz

$$\phi(t,z,r) = \phi_0(p)e^{-i\omega t+iqz} f_p(r)\,, \qquad p^2 \equiv -\omega^2 + q^2$$

$$0 = \left(\frac{f_p'(\hat{x})}{\hat{x}^3}\right)' - \frac{p^2}{\hat{x}^3} f_p(\hat{x})\,, \qquad \hat{x} \equiv \frac{L^2}{r}$$

- The general solution (for space-like $p^2 > 0$ momentum) takes form

$$f_p = (p^2\hat{x}^2)\bigg(C_1\ I_2(p\hat{x}) + C_2\ K_2(p\hat{x})\big)\bigg)$$

where C_i are the integration constants. Requiring that f_p is finite everywhere, and is normalized at the boundary (without the loss of generality) as

$$\lim_{\hat{x}\to 0} f_p = 1$$

we conclude

$$f_p = \frac{1}{2}(p^2\hat{x}^2)\ \ K_2(p\hat{x})$$

- The bulk action S_{SUGRA} evaluated on the solution can be integrated by parts:

$$S_{SUGRA} = \frac{N^2}{8\pi^2 L^3} \int d^5x\sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right)$$

$$= \frac{N^2}{16\pi^2 L^3} \int d^5x\sqrt{-g}\phi\Box\phi - \frac{N^2}{16\pi^2 L^3} \int d^4x \frac{r^3}{L^3}\frac{r^2}{L^2}\phi(x,r)\partial_r\phi_(x,r)\Big|_{r\to\infty}$$

$$= \frac{N^2}{16\pi^2} \int d^4x \frac{1}{\hat{x}^3}\phi(x,\hat{x})\partial_{\hat{x}}\phi(x,\hat{x})\Big|_{\hat{x}\to 0}$$

$$= \int \frac{d^4p}{(2\pi)^4}\phi_0(-p)\mathcal{F}(p,\hat{x})\phi_0(p)\Big|_{\hat{x}\to 0}$$

with

$$\mathcal{F}(p,\hat{x}) = \frac{N^2}{16\pi^2}\frac{1}{\hat{x}^3} f_{-p}(\hat{x})\partial_{\hat{x}} f_p(\hat{x})$$

where in the third line we used EOM; and in the last line we rewrote the integral in Fourier space.

- Note

$$G(x-y) = -\frac{\delta^2 S_{SUGRA}}{\delta J(x)\delta J(y)} \implies S_{SUGRA} = -\frac{1}{2}\int d^4x d^4y G(x-y)J(x)J(y)$$

or in Fourier space

$$S_{SUGRA} = \int \frac{d^4p}{2\pi)^4} J(p)\Big(-\frac{1}{2}G(p)\Big)J(-p)$$

- In our case

$$J(p) \equiv \phi_0(p)$$

thus

$$G(p) = -2 \lim_{\hat{x}\to 0} \mathcal{F}(p,\hat{x})$$

- As $\hat{x} \to 0$, and for space-like momenta ($p^2 > 0$)

$$f_p(\hat{x}) = 1 - \frac{1}{4}(p\hat{x})^2 - \frac{1}{16}(p\hat{x})^4 \ln(p\hat{x}) + \cdots$$

thus,

$$G(p) = \frac{N^2}{64\pi^2} p^4 \ln(p^2)$$

- Note that in evaluating $G(p)$ we dropped $\sim p^2\hat{x}^{-2}$ and $\sim p^4\ln\hat{x}$ divergent terms. These terms can be removed by the holographic renormalization procedure where one introduces finite counterterms in evaluating the correlation function.

$\Longrightarrow$ Identical steps can be repeated for the computation of the thermal correlation function. I will review now the important parts, and highlight the differences.

- The unperturbed metric now is that of the AdS-BH:

$$ds_5^2 = \frac{r^2}{L^2}\left(-f dt^2 + (d\vec{x})^2\right) + \frac{L^2}{r^2 f}dr^2\,, \qquad f = 1 - \frac{r_0^2}{r^2}$$

or using $u \equiv \frac{r_0^2}{r^2}$ coordinate

$$ds_5^2 = \frac{(\pi T L)^2}{u}(-f(u)dt^2 + (d\vec{x})^2) + \frac{L^2}{4u^2 f(u)}du^2\,, \qquad f(u) = 1 - u^2$$

- The fluctuation

$$\phi = h_y^x = \phi_0(p) e^{-i\omega t + iqz} f_p(u) \,, \qquad p = (w, 0, 0, q)$$

decouples from the rest, and satisfied the EOM of the massless minimally coupled scalar in the unperturbed background:

$$f_p'' - \frac{1+u^2}{uf} \; f_p' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2} \; f_p = 0$$

where

$$\mathfrak{w} = \frac{\omega}{2\pi T} \,, \qquad \mathfrak{q} = \frac{q}{2\pi T}$$

- Previously, we required the regularity of f_p everywhere in AdS and normalization to unity as one approaches the boundary,

$$f_p(u \to 0) = 1$$

- There is additional subtlety associated with the choice of the boundary condition near the horizon. Assuming the ansatz

$$f_p = (1-u)^\alpha \,, \qquad u \to 1_-$$

we find (plugging the ansatz into the equation)

$$0 = \alpha^2 + \frac{\mathfrak{w}^2}{4}\,, \qquad \Longrightarrow \qquad \alpha_\pm = \pm\frac{i}{2}\mathfrak{w}$$

- Thus, near the horizon

$$\phi \sim exp\left(-i\omega t \pm \frac{i}{2}\mathfrak{w}\ln(1-u)\right) = exp\left(-i\omega\left(t \mp \frac{1}{4\pi T}\ln(1-u)\right)\right)$$

  That is:

  - mode with $\alpha = \alpha_-$ moves **into** the horizon (is in-falling)
  - mode with $\alpha = \alpha_+$ moves **out of** the horizon (is out-going)

- Son and Starinets argued that in-falling boundary conditions correspond to computation of the **retarded** correlation function, while the outgoing boundary condition correspond to computation of the **advanced** correlation function. Moreover, the relation between the kernel

$$\mathcal{F}^{R/A}(p,u) = \frac{N^2}{16\pi^2 L^3}\sqrt{-g}g^{uu}\; f_{-p}^{R/A}(u)\;\partial_u f_p^{R/A}(u)$$

  is exactly the same as for the zero temperature case:

$$G^{R/A}(p) = -2\lim_{u\to 0}\mathcal{F}^{R/A}(p,u)$$

  (up to divergent contact terms)

- For concreteness, consider retarded correlation function. We can not solve the equation for f_p for general p, but we can solve it perturbatively in $\mathfrak{w}$ for $\mathfrak{q}=0$. Indeed, assuming

$$f_p = (1-u)^{-i\mathfrak{w}/2}\left(F_0(u) + i\ \mathfrak{w} F_1(u) + \mathcal{O}(\mathfrak{w}^2)\right)$$

we find the following equations

$$0 = F_0'' + \frac{u^2+1}{u(u^2-1)}\ F_0'$$

$$0 = F_1'' + \frac{u^2+1}{u(u^2-1)}\ F_1' + \frac{1}{1-u}\ F_0' + \frac{1}{2u(u^2-1)}\ F_0$$

- Above equation have to be solved, subject to analyticity at the horizon and

$$F_0(u\to 0) = 1\,, \qquad F_1(u\to 0) = 0$$

- There is a unique solution with prescribed boundary conditions:

$$F_0(u) = 1\,, \qquad F_1(u) = -\frac{1}{2}\ln(u+1)$$

  leading to

$$\mathcal{F}^R(\mathfrak{w}, u) = \frac{\pi^2 N^2 T^4}{8}\; i\mathfrak{w} + \mathcal{O}(\mathfrak{w}^2)$$

  Note that there are no divergences here to order $\mathcal{O}(\mathfrak{w})$.

- Finally,

$$G^R(\mathfrak{w}) = -\frac{\pi^2 N^2 T^4}{4}\; i\mathfrak{w} + \mathcal{O}(\mathfrak{w}^2)$$

  and using the Kubo formula,

$$\eta = -\lim_{\omega\to 0}\frac{1}{2\pi T\mathfrak{w}}\operatorname{Im} G^R(\mathfrak{w}) = \frac{\pi N^2 T^3}{8}$$

- Recalling the entropy density of the strongly coupled $\mathcal{N} = 4$ SYM

$$s = \frac{\pi^2 N^2 T^3}{2}$$

we arrive at the famous Policastro-Son-Starinets result:

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Gauge invariant fluctuations in $\mathcal{N} = 4$ SYM plasma

$\Longrightarrow$ We now would like to compute the spectrum of hydrodynamic modes in plasma, from the holographic perspective.

$\Longrightarrow$ Following extended AdS/CFT (gauge/gravity) dictionary the **physical modes in plasma** are dual to **quasinormal modes in BH geometry**

$\Longrightarrow$ Recall again, the equilibrium state of $\mathcal{N} = 4$ plasma is described by a BH solution:

$$ds_5^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{\pi^2 T^2}{u} \left(-(1-u^2) dt^2 + d\vec{x}^3 \right) + \frac{du^2}{4(1-u^2)u^2}$$

where T is the Hawking temperature of the BH (to be identified with the plasma temperature)

$\Longrightarrow$ Consider the graviton $h_{\mu\nu}$ fluctuations

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$$

$\Longrightarrow$ We can always choose the gauge

$$h_{\mu u} = 0$$

$\Longrightarrow$ At a linearized level, we can assume that

$$h_{\mu\nu} = h_{\mu\nu}(t, x_3, u) \sim e^{-i\omega t + iqx_3}$$

$\Longrightarrow$ Above fluctuations preserve $O(2)$ symmetry — rotations in $x_1 - x_2$ plane

$\Longrightarrow$ Because of the symmetry, fluctuations of different helicities would decouple from each other:

$$\text{helicity} - 2: \qquad \{h_{x_1x_2}\}\,,\ \{h_{x_1x_1} - h_{x_2x_2}\}$$

$$\text{helicity} - 1: \qquad \{h_{tx_1},\ h_{x_1x_3}\}\,,\ \{h_{tx_2},\ h_{x_2x_3}\}$$

$$\text{helicity} - 0: \qquad \{h_{tt},\ h_{aa} \equiv h_{x_1x_1} + h_{x_2x_2},\ h_{tx_3},\ h_{x_3x_3}\}$$

$\Longrightarrow$ The shear modes correspond to helicity-1 fluctuations; the sound modes are encoded in helicity-0 fluctuations; helicity-2 fluctuations are not hydrodynamic (as we saw earlier they can be used to compute shear viscosity via Kubo formula)

$\Longrightarrow$ Consider helicity-0 modes. Introduce

$$h_{tt} = e^{-i\omega t + iqx_3}\ \frac{\pi^2 T^2 (1-u^2)}{u}\ H_{tt}(u)\,, \qquad h_{tz} = e^{-i\omega t + iqx_3}\ \frac{\pi^2 T^2}{u}\ H_{tz}(u)$$

$$h_{aa} = e^{-i\omega t + iqx_3}\ \frac{\pi^2 T^2}{u}\ H_{aa}(u)\,, \qquad h_{zz} = e^{-i\omega t + iqx_3}\ \frac{\pi^2 T^2}{u}\ H_{zz}(u)$$

From the Einstein equations

$$R_{\mu\nu}\left[g_{\mu\nu} + h_{\mu\nu}\right] = -4(g_{\mu\nu} + h_{\mu\nu})$$

we obtain

- 4 second-order differential equations for

$$\{H_{tt}, H_{tz}, H_{aa}, H_{zz}\}$$

- 3 first order differential constraints associated with fixing the gauge

$$h_{tu} = h_{zu} = h_{uu} = 0$$

$\Longrightarrow$ Had the constraints been algebraic, it could have been used to eliminate 3 fields, and produce a single (4-3=1) second order differential equation

$\Longrightarrow$ The differential elimination is possible if one uses a gauge-invariant variables! (Kovtun-Starinets):

- first, identify residual diffeomorphisms

$$x^\mu \to x^\mu + \xi^\mu \qquad \Rightarrow \qquad g_{\mu\nu} \to g_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$$

such that

$$g_{\mu u} \to g_{\mu u} \qquad \Rightarrow \qquad 0 = \nabla_\mu \xi_r + \nabla_r \xi_\mu$$

- under above transformations

$$\{H_{tt}, H_{tz}, H_{aa}, H_{zz}\} \to \{H'_{tt}, H'_{tz}, H'_{aa}, H'_{zz}\}$$

- second, introduce a linear combination of metric fluctuations that stays invariant

$$Z \equiv 4\frac{q}{w} H_{tz} + 2H_{zz} - H_{aa}\left(1 - (1+u^2)\frac{q^2}{w^2}\right) + 2(1-u^2)\frac{q^2}{w^2} H_{tt} \to \tilde{Z} = Z$$

$\Longrightarrow$ The equation of motion for Z can be completely decoupled:

$$0 = Z'' + \mathcal{A}_z\ Z' + \mathcal{B}_z\ Z$$

$$\mathcal{A}_z = -\frac{3\mathfrak{q}^2u^4 - 2\mathfrak{q}^2u^2 + 3\mathfrak{q}^2 - 3\mathfrak{w}^2u^2 - 3\mathfrak{w}^2}{(-1+u^2)u(-3\mathfrak{q}^2 + 3\mathfrak{w}^2 + \mathfrak{q}^2u^2)}$$

$$\mathcal{B}_z = \frac{4\mathfrak{q}^2u^5 + \mathfrak{q}^4u^4 - 4\mathfrak{q}^2u^3 + 4\mathfrak{w}^2\mathfrak{q}^2u^2 - 4\mathfrak{q}^4u^2 + 3\mathfrak{q}^4 + 3\mathfrak{w}^4 - 6\mathfrak{w}^2\mathfrak{q}^2}{u(-3\mathfrak{q}^2 + 3\mathfrak{w}^2 + \mathfrak{q}^2u^2)(-1+u^2)^2}$$

where

$$\mathfrak{w} = \frac{w}{2\pi T}\,, \qquad \mathfrak{q} = \frac{q}{2\pi T}$$

$\Longrightarrow$ Let's analyze the asymptotic behavior of above equation near the horizon, i.e., as $u \to 1$

$$Z \sim (1-u^2)^{\alpha}\,, \qquad \Rightarrow \qquad \alpha = \pm i\frac{\mathfrak{w}}{2}$$

Thus, near the horizon,

$$z(t, x_3, u) = e^{-iwt+iqx^3} Z(u) \sim \exp\left[-i\mathfrak{w}\left(2\pi T t \mp \frac{1}{2}\ln(1-u)\right) + iqx_3\right]$$

So, the modes with $\alpha = -i\frac{\mathfrak{w}}{2}$ moves *into the horizon* and modes with $\alpha = +i\frac{\mathfrak{w}}{2}$ moves *away from the horizon*

$\Longrightarrow$ Near the boundary, $u \to 0$,

$$Z \sim \#\, 1 + \# u^2$$

The leading asymptotic actually changes the background metric (we varied it earlier for the Kubo formula), thus, to determine physical fluctuations in $\mathcal{N} = 4$ SYM plasma in flat space-time we must insist that

$$Z(u \to 0) = 0$$

leading to a *Dirichlet* condition at the boundary

$\Longrightarrow$ Notice that equation for Z is homogeneous, so imposing

$$\alpha = +i\frac{\mathfrak{w}}{2} \qquad + \qquad \textit{Dirichlet}$$

or

$$\alpha = -i\frac{\mathfrak{w}}{2} \qquad + \qquad \textit{Dirichlet}$$

would determine the dispersion relation for the quasinormal mode Z:

$$\mathfrak{w} = \mathfrak{w}(\mathfrak{q})$$

A careful analysis of the quasinormal equation show that

$$\alpha = \pm i\frac{\mathfrak{w}}{2} \qquad \Rightarrow \qquad \pm\mathrm{Im}\left[\mathfrak{w}(\mathfrak{q})\right] > 0$$

$\Longrightarrow$ Poles in the retarded (advanced) correlation function of the stress-energy tensor correspond to the gravitational fluctuations with $\alpha = -i\frac{\mathfrak{w}}{2}$ $(\alpha = +i\frac{\mathfrak{w}}{2})$

$\Longrightarrow$ It is not possible to solve equation for Z analytically; in the hydrodynamic limit

$$\mathfrak{w} \ll 1\,, \qquad \mathfrak{q} \ll 1\,, \qquad \mathfrak{w} \sim \mathfrak{q}$$

we find (up to an overall constant)

$$Z(u) = (1-u^2)^{-i\mathfrak{w}/2}\left(z_0(u) + i\;\mathfrak{w}\;z_1(u) + \mathcal{O}(\mathfrak{w}^2, \mathfrak{q}^2)\right)$$

with

$$z_0 = \frac{\mathfrak{q}^2(1+u^2) - 3\mathfrak{w}^2}{2\mathfrak{q}^2 - 3\mathfrak{w}^2}\,, \qquad z_1 = \frac{2\mathfrak{q}^2(u^2-1)}{2\mathfrak{q}^2 - 3\mathfrak{w}^2}$$

The Dirichlet boundary condition $Z(0) = 0$ then determines the *sound channel* quasinormal (hydrodynamic) mode:

$$\mathfrak{w} = \pm\frac{1}{\sqrt{3}}\mathfrak{q} - \frac{i}{3}\mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3)$$

Comparing with the hydro prediction we read-off

$$c_s^2 = \frac{1}{3}\,, \qquad 2\pi T\Gamma = \frac{1}{3}\;4\pi\frac{\eta}{s}\;\left(1 + \frac{3}{4}\frac{\zeta}{\eta}\right) = \frac{1}{3}$$

for a conformal theory $\frac{\zeta}{\eta} = 0 \Longrightarrow$

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

There are number of important consequences of this computation:

- Note that c_s^2 computed from the graviton fluctuations precisely matches the value that can be obtained from the equilibrium thermodynamics, using equation of state:

$$c_s^2 = \frac{\partial P}{\partial \epsilon} = \frac{1}{3}$$

- The shear viscosity computed from the sound channel fluctuations precisely matches the result from the Kubo formula. Note that (what I hid under the rug) this agreement provides a (often a highly) nontrivial check on holographic renormalization, that should be used carefully compute the correlation functions.

Of course, all above consistency checks are **mandatory to be true** once we declare that hydrodynamic description of the system is consistent.

What is remarkable that this 4d hydro consistencies are **true** in dual 5d gravitational system. Such checks were the first indication that gauge/gravity correspondence is true in nonequilibrium setting. Later this was developed in its full glory and is known now as *gravity-fluid* correspondence

$\Longrightarrow$ Later I will mention that consistency checks extend beyond the supergravity approximation in the gauge/string correspondence.

$\Longrightarrow$ Consider helicity-1 modes. Introduce

$$h_{tx} = e^{-i\omega t + iqx_3} \, \frac{\pi^2 T^2}{u} \, H_{tx}(u) \,, \qquad h_{xz} = e^{-i\omega t + iqx_3} \, \frac{\pi^2 T^2}{u} \, H_{xz}(u)$$

From the Einstein equations we obtain

- 2 second-order differential equations for

$$\{H_{tx}, H_{xz}\}$$

- 1 first order differential constraints associated with fixing the gauge

$$h_{xu} = 0$$

$\Longrightarrow$ Again, differential elimination of the constraint via diffeo-invariant variables implies that there should be (2-1=1) second order 'physical' differential equation

Indeed,

$$Z_{shear} \equiv qH_{tx} + \omega H_{xz} \to \tilde{Z}_{shear} = Z_{shear}$$

stays invariant under the all residual diffeomorphisms.

$\Longrightarrow$ The equation of motion for Z_{shear} can be completely decoupled:

$$0 = Z''_{shear} + \frac{(x^2\mathfrak{q}^2 + \mathfrak{w}^2)}{x(\mathfrak{w}^2 - x^2\mathfrak{q}^2)}\ Z'_{shear} + \frac{\mathfrak{w}^2 - x^2\mathfrak{q}^2}{x^2(1-x^2)^{3/2}}\ Z_{shear}$$

where

$$x \equiv \left(1 - u^2\right)^{1/2}$$

The incoming boundary condition at the horizon $(x \to 0^+)$ implies that

$$Z_{shear}(x) = x^{-i\mathfrak{w}} z_{shear}(x)\,,$$

where $z_{shear}(x)$ is regular at the horizon.

Without loss of generality we can assume

$$z_{shear}\Big|_{x\to 0^+} = 1\,,$$

the spectrum of quasinormal frequencies is then determined by imposing a Dirichlet condition at the boundary

$$z_{shear}\Big|_{x\to 1^-} = 0$$

In the hydrodynamic approximation ($\mathfrak{w} \ll 1$ and $\mathfrak{q} \ll 1$) the solution can be written in the ansatz

$$z_{shear} = z^{(0)}_{shear} + i\ \mathfrak{q} z^{(1)}_{shear} + \mathcal{O}(\mathfrak{q}^2)$$

Substituting the ansatz into EOM, and we find

$$z^{(0)}_{shear} = 1\,, \qquad z^{(1)}_{shear} = \frac{1}{2}\frac{\mathfrak{q}}{\mathfrak{w}}\ x^2\,,$$

Imposing the Dirichlet boundary conditions determines the lowest shear quasinormal frequency as

$$\mathfrak{w} = -i\ \frac{1}{2}\ \mathfrak{q}^2$$

Note

$$\mathfrak{w} = -i\ \frac{1}{2}\ \mathfrak{q}^2\,, \qquad \Longrightarrow \qquad \omega = -i\frac{q^2}{T}\ \frac{1}{4\pi}$$

which matching with the hydrodynamics implies

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Once again, there is a consistency in the quasinormal spectrum of translationary invariant AdS BH with that of the hydrodynamics of 4d relativistic fluid.

Shear viscosity bound

Similar analysis can be performed in generic (infinitely) strongly coupled (planar) gauge theory

$$\frac{1}{Ng_{YM}^2} \to 0 \qquad \& \qquad N \to \infty \qquad (\text{with } Ng_{YM}^2 \to \text{const })$$

We find:

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

for:

- any <u>product of gauge groups</u> and matter content in arbitrary <u>representations</u>
- arbitrary <u>non-conformal</u> deformations (masses, non-zero β-functions)
- arbitrary <u>chemical potentials</u> for conserved $U(1)$'s
- <u>non-commutativity</u> of the background of space-time
- presence of (scalar) <u>Goldstone modes</u> from spontaneous breaking of continuous symmetries (superfluid/superconductor)
- background <u>electromagnetic</u> fields

$\Longrightarrow$ A theorem can be proven that in isotropic strongly coupled plasma (under wide range of conditions) (see arXiv:hep-th/0610145)

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

I am not going to prove it here, but provide some steps:

- the proof follows the Kubo formula result for $\mathcal{N} = 4$ SYM plasma
- one assumes a 5d low-energy effective action containing scalars and vectors describing the gravitational dual of the plasma
- one argues that the metric fluctuation

$$h_{\mu\nu} = h_{\mu\nu}(t, z, x)$$

with indexes $(\mu\nu) = (xy)$ decouples from all the other fluctuations, and behaves as a minimally coupled scalar in hairy BH background of the type

$$ds_5^2 = c_1(x)^2(-(1-x)^2dt^2 + (d\vec{x})^2) + c_2(x)^2\ dx^2$$

Note that I wrote a BH metric using a universal radial coordinate x.

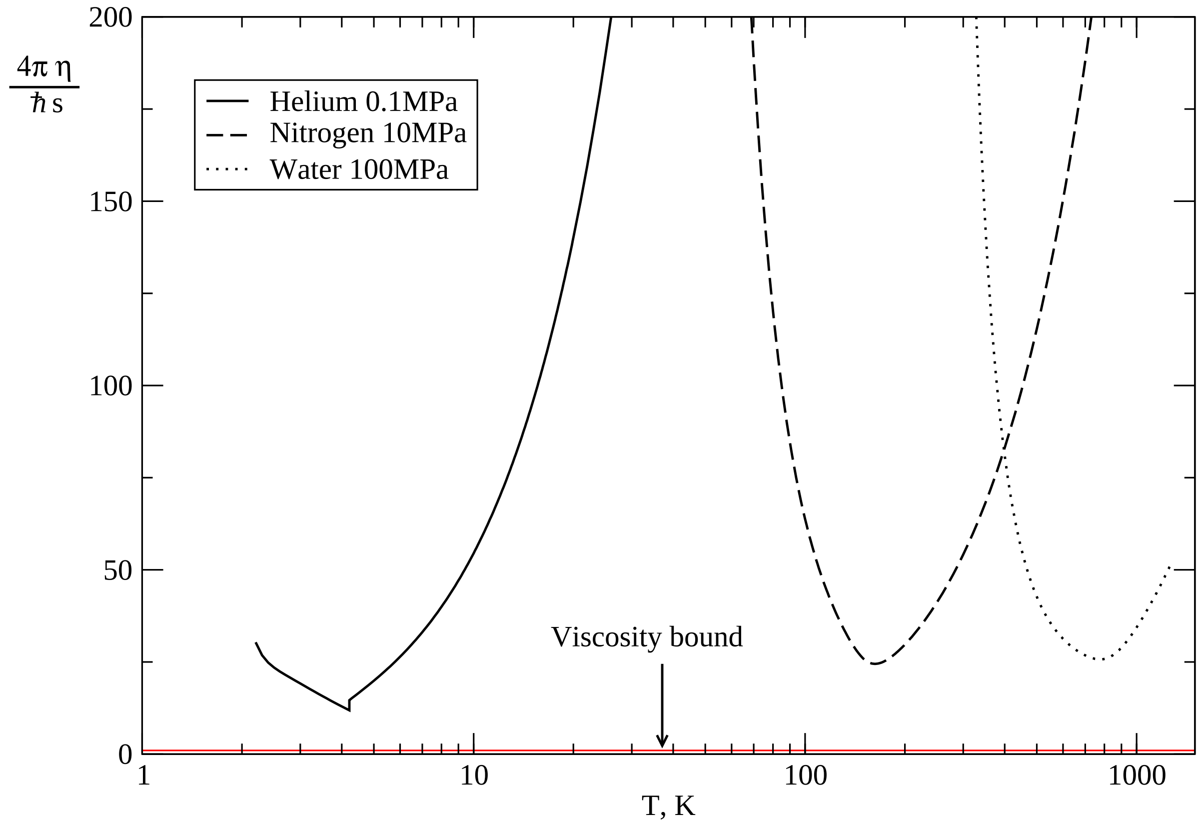
- The equation of motion for

$$\phi \equiv h_x^y = e^{-i\omega t+iqz} f_p(x)$$

can be solved analytically in the hydrodynamic limit (to order $\mathcal{O}(\mathfrak{w})$ and at $\mathfrak{q}=0$) in the universal x radial coordinate.

- We can compute the shear viscosity from the Kubo formula via 2-point retarded correlation function for the T_{xy}, as in $\mathcal{N}=4$ SYM case; the result depends on a parameter which is (universally) proportional to the entropy density.
- The ratio $\frac{\eta}{s}$ comes naturally from the computation

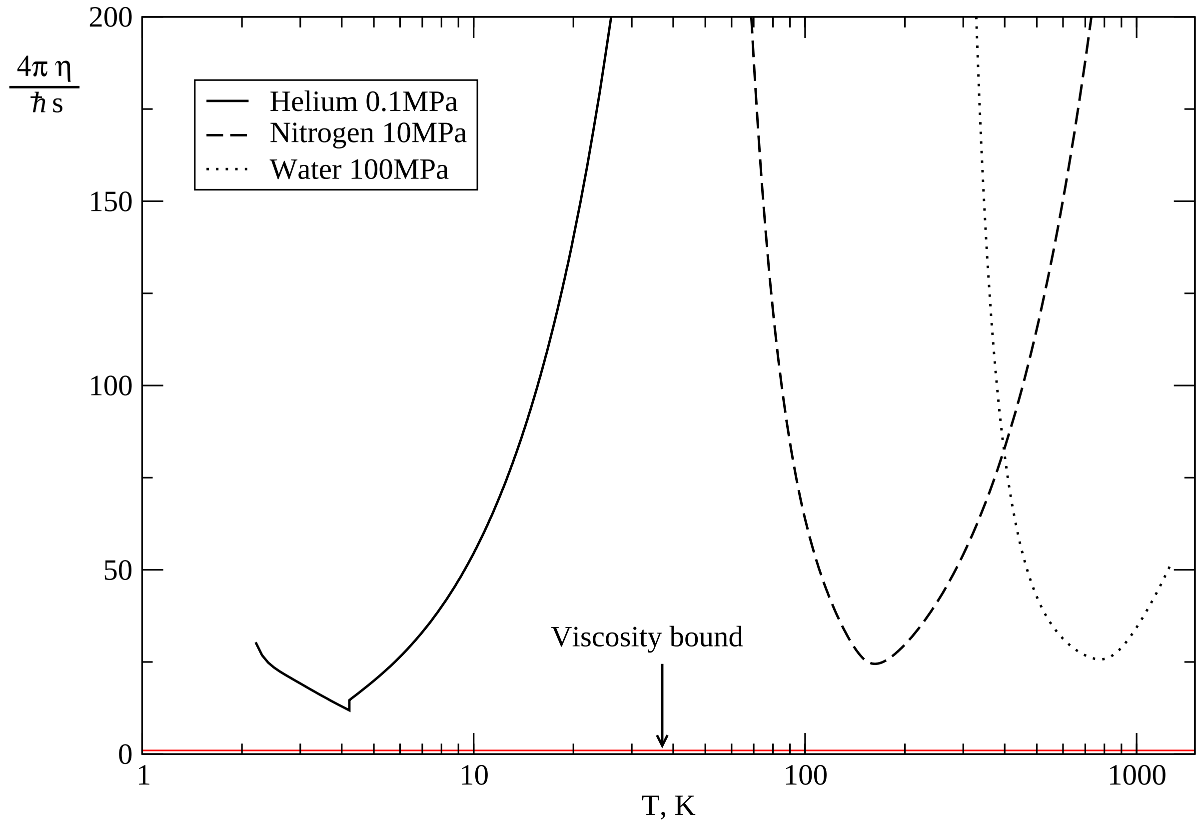
KSS viscosity bound



From: P.Kovtun, D.T.Son, A.O.Starinets, Phys.Rev.Lett. 94 (2005) 111601

So, is there really a bound for the shear viscosity to the entropy density?

KSS viscosity bound



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So, is there really a bound for the shear viscosity to the entropy density?

Stay tunes for the last lecture!

Holographic non-conformal hydrodynamics

$\Longrightarrow$ As long as translational invariance in plasma is unbroken, there is not much to be added to the shear viscosity because of the universality. Couple points though:

- Presence of matter, chemical potentials for the global conserved $U(1)$ charges might produce a critical thermodynamic behaviour (recall $\mathcal{N} = 2^*$ plasma). As a result, the entropy density can have interesting scaling close to criticality as well.
- Because of the universality for the ratio $\frac{\eta}{s}$, the shear viscosity at strong coupling is guaranteed to follow the critical behaviour of the entropy density.

$\Longrightarrow$ The main motivation to study sound waves in nonconformal plasma (in my view) is to get a handle on the bulk viscosity. Remarkably, computation of the bulk viscosity is so complicated even at weak coupling, that the first reliable analysis of this quantity was done in holography!

Analysis of the non-conformal model, although technically more difficult, are conceptually identical

$\Longrightarrow$ Consider $\mathcal{N} = 2^*$ model:

- There is a decoupled set of helicity-0 fluctuations in the background, dual to a sound wave,

$$\{H_{tt}, H_{tz}, H_{aa}, H_{zz}\} + \{\delta\alpha, \delta\chi\}$$

where $\{\delta\alpha\}$ and $\{\delta\chi\}$ are the fluctuations of the background supergravity scalars $\{\alpha, \chi\}$

- we expect 4+2-3=3 independent coupled second-order equations for the fluctuations
- for the metric ansatz

$$ds_5^2 = -c_1^2(r)\ dt^2 + c_2^2(r)\ d\vec{x}^2 + dr^2$$

the gauge-invariant combinations of the fluctuations are:

$$Z_H = 4\frac{q}{\omega}\ H_{tz} + 2\ H_{zz} - H_{aa}\left(1 - \frac{q^2}{\omega^2}\frac{c_1' c_1}{c_2' c_2}\right) + 2\frac{q^2}{\omega^2}\frac{c_1^2}{c_2^2}\ H_{tt}$$

$$Z_\phi = \delta\alpha - \frac{\alpha'}{(\ln c_2^4)'}\ H_{aa}$$

$$Z_\psi = \delta\chi - \frac{\chi'}{(\ln c_2^4)'}\ H_{aa}$$

and the equations take form:

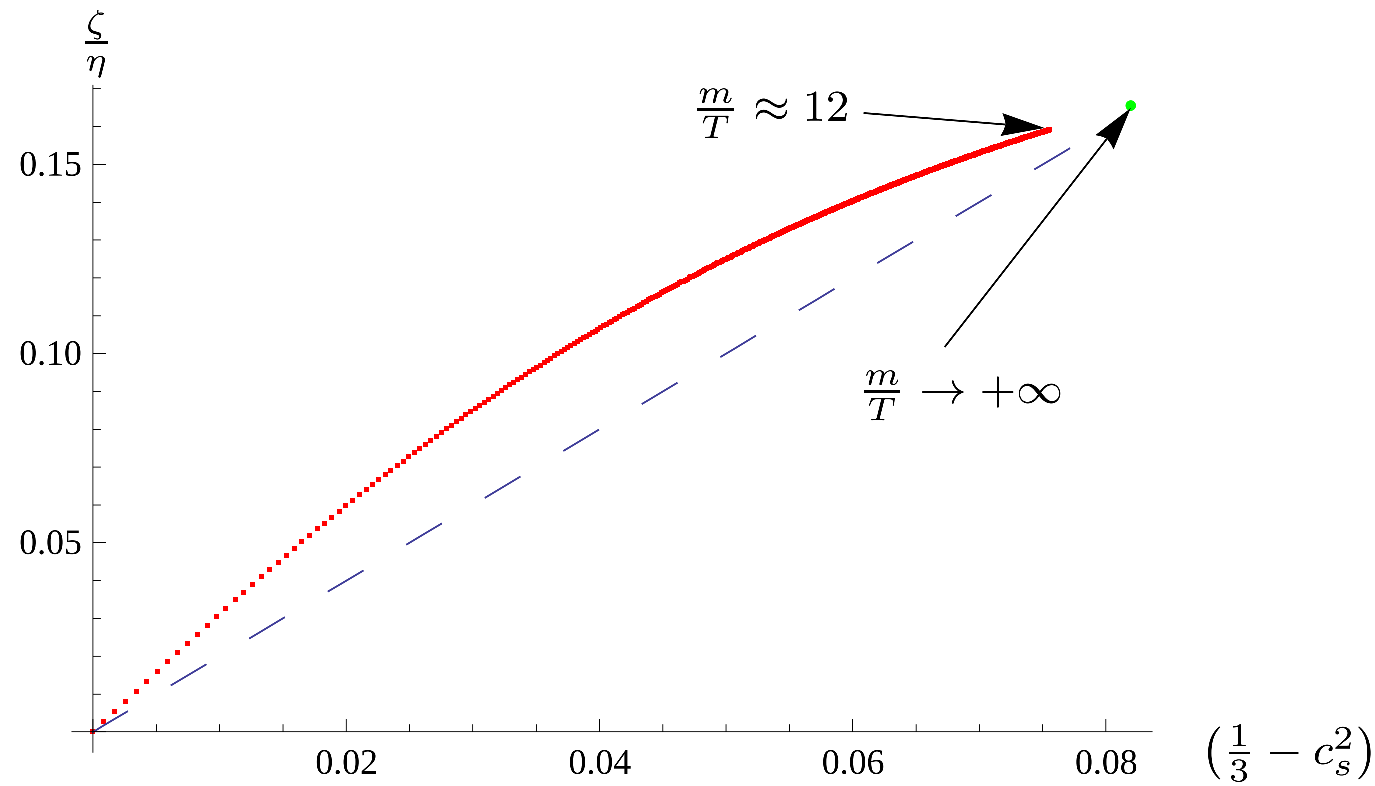
$$A_H\, Z_H'' + B_H\, Z_H' + C_H\, Z_H + D_H\, Z_\phi + E_H\, Z_\psi = 0$$

$$A_\phi Z_\phi'' + B_\phi Z_\phi' + C_\phi Z_\phi + D_\phi Z_\psi + E_\phi Z_H' + F_\phi Z_H = 0$$

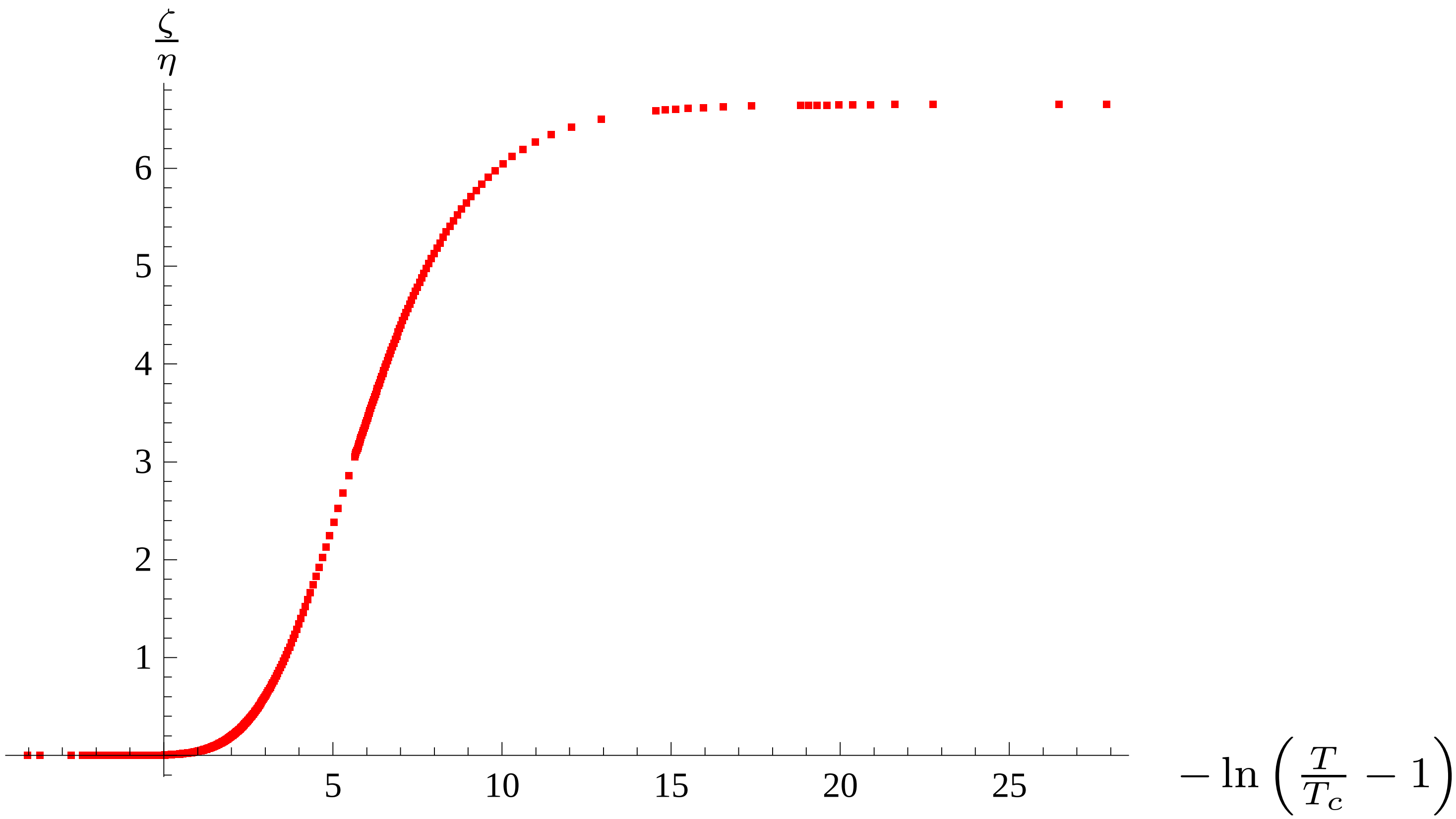
$$A_\psi Z_\psi'' + B_\psi Z_\psi' + C_\psi Z_\psi + D_\psi Z_\phi + E_\psi Z_H' + F_\psi Z_H = 0$$

where the coefficients $\{A..., B..., \cdots, F...\}$ depend on the background values c_1, c_2, α, χ.

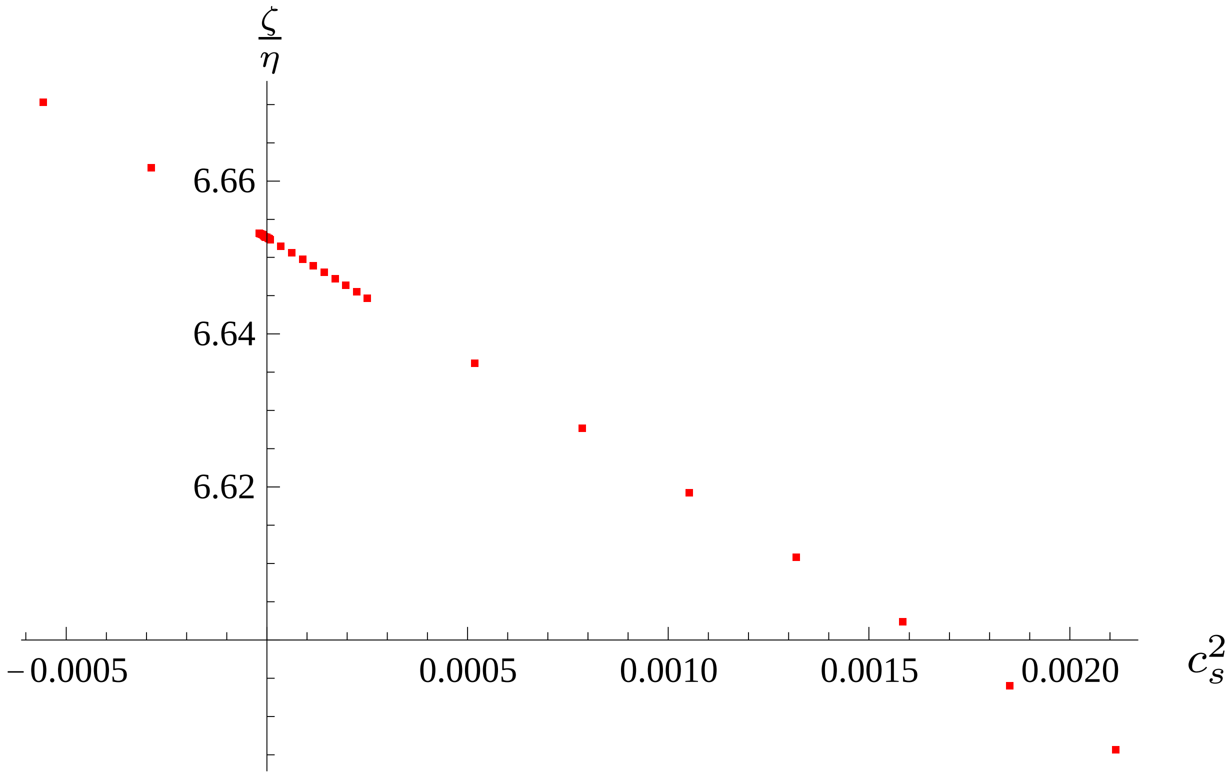
$\Longrightarrow$ The rest of analysis goes as in $\mathcal{N} = 4$ case, except numerically.



Ratio of viscosities $\frac{\zeta}{\eta}$ versus the speed of sound in $\mathcal{N} = 2^*$ gauge theory plasma with "supersymmetric" mass deformation parameters $m_b = m_f = m$. The dashed line represents the bulk viscosity inequality $\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - c_s^2\right)$. We computed the bulk viscosity up to $m/T \approx 12$. A single point represents extrapolation of the speed of sound and the viscosity ratio to $T \to +0$.



Ratio of viscosities $\frac{\zeta}{\eta}$ in $\mathcal{N}=2^*$ gauge theory plasma with zero fermionic mass deformation parameter $m_f=0$.



Ratio of viscosities $\frac{\zeta}{\eta}$ in $\mathcal{N} = 2^*$ gauge theory plasma near the critical point. Note that the critical point corresponds to $c_s^2 = 0$.

$\Longrightarrow$ Notice that the bulk viscosity is finite at the mean-field-theory critical point.

Estimates for the viscosity of QGP at RHIC. It is tempting to use the $\mathcal{N}=2^*$ strongly coupled gauge theory plasma results to estimate the bulk viscosity of QGP produced at RHIC. For c_s^2 in the range $0.27-0.31$, as in QCD at $T=1.5T_{deconfinement}$ we find

$$\frac{\zeta}{\eta}\bigg|_{m_f=0}\approx 0.17-0.61\,,\qquad \frac{\zeta}{\eta}\bigg|_{m_b=m_f=m}\approx 0.07-0.15\,. \tag{1}$$

Since RHIC produces QGP close to its criticality, we believe that $m_f=0$ $\mathcal{N}=2^*$ gauge theory model would reflect physics more accurately.

$\Longrightarrow$ Computation of the sound channel quasinormal modes also allows for a comparison between the speed of sound directly extracted from the sound-wave dispersion relation, and that of computed from the equilibrium thermodynamics, assuming applicability of relativistic hydrodynamics.

$\Longrightarrow$ I would like to present results of two computations:

- sound waves in cascading gauge theory plasma
- sound in $\mathcal{N} = 4$ SYM with $U(1)_R$ chemical potential, deformed by a (small) mass term to chiral multiples

$\Longrightarrow \mathcal{N}=1$ cascading gauge theory plasma: $SU(P+K_\star)\times SU(K_\star)$ with

$$K_\star \propto 2P^2\ \ln\frac{T}{\Lambda}\,, \qquad T\gg\Lambda$$

(high temperature). Here,

$$c_s^2\Big|_{high-T}=\frac13\biggl\{1-\frac43\frac{P^2}{K_\star}+\left(2\beta_{1,2}+\frac49\right)\frac{P^4}{K_\star^2}+\left(2\beta_{1,3}-\frac43\beta_{1,2}\right)\frac{P^6}{K_\star^3}$$
$$+\left(2\beta_{1,4}-\frac43\beta_{1,3}+(\beta_{1,2})^2\right)\frac{P^8}{K_\star^4}+\mathcal{O}\left(\frac{P^{10}}{K_\star^5}\right)\biggr\}$$

n	1	2	3	4
$\left(1-\frac{\beta_{1,n}^{thermo}}{\beta_{1,n}^{sound}}\right)$	0	-1.8×10^{-8}	4.0×10^{-7}	5.5×10^{-7}

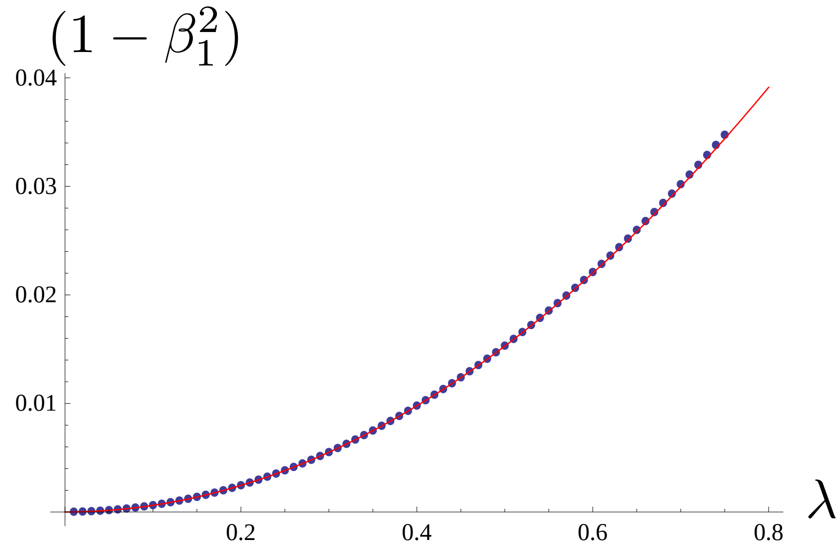
where

thermo – from equilibriumthermodynamics

sound – from spectrum of quasinormal modes

$\Longrightarrow \mathcal{N}=4$ SYM plasma at infinite 't Hooft coupling at finite chemical potential μ, perturbed by a relevant operator (chiral multiple mass term) to leading order in $\frac{M}{T}$, $\frac{M}{\mu}$

$$c_s^2 = \left((\epsilon + P)\ \frac{\partial(P,\rho)}{\partial(T,\mu)} + \rho\ \frac{\partial(\epsilon,P)}{\partial(T,\mu)} \right) \left((\epsilon + P)\ \frac{\partial(\epsilon,\rho)}{\partial(T,\mu)} \right)^{-1}$$



where

$$\beta_1 = 3c_s^2, \lambda = \frac{(\kappa+2)}{2^{3/4}(1+\kappa)^{3/4}\pi}\ \frac{M}{T} \left(1 + \mathcal{O}\left(\frac{M^2}{T^2} \right) \right)\ , \frac{2\pi T}{\mu} = \sqrt{\kappa} + \frac{2}{\sqrt{\kappa}}$$

$\Longrightarrow$ Red line from thermodynamics; blue dots from the spectrum

Bulk viscosity bound

- In lots of explicit examples of gauge-string duality, for a strongly coupled plasma d spatial dimensions,

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{d} - c_s^2\right)$$

However, there are also holographic examples where the bound is being violated:

- some phenomenological models of gauge/gravity correspondence
- $\mathcal{N} = 4$ SYM compactified on $\mathcal{M}_2$ of constant negative curvature
- Bulk viscosity is finite in the vicinity of the phase transition; but is grows very rapidly:

$$\frac{d\zeta}{dT} \sim T_c^2 \left(1 - \frac{T_c}{T}\right)^{-1/2}, \qquad T \to T_c$$

$\Longrightarrow$ I want to show how holographic is useful to exclude models of critical behaviour of bulk viscosity

Recall:

$$T^{\mu\nu} = T^{\mu\nu}_{equilibrium} + T^{\mu\nu}_{non-equilibrium}$$

$$T^{\mu\nu}_{eq} = \epsilon\ u^\mu u^\nu + P\Delta^{\mu\nu}\,, \qquad T^{\mu\nu}_{non-eq} = -\eta\ \sigma^{\mu\nu} - \zeta(\nabla u)$$

$$u^\mu u_\mu = -1\,, \qquad \Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$$

where η, ζ are the shear and the bulk viscosities and $\sigma^{\mu\nu}$ is a shear tensor (which is traceless):

$$\eta_{\mu\nu}\sigma^{\mu\nu} = 0$$

In CFT $\qquad T^\mu_\mu = 0 \qquad \Longrightarrow$

$$-\epsilon + 3P - 3\zeta(\nabla u) = 0 \qquad \Longrightarrow \qquad \epsilon = 3P\Big|_{CFT}\,, \qquad \zeta\Big|_{CFT} = 0$$

$\Longrightarrow$ so in order to see $\zeta \neq 0$ we need to look @ non-conformal theories

- Naively, second-order phase transitions imply scale invariance $\Longrightarrow$

$$\zeta \to 0 \qquad \text{or} \qquad \zeta \to \infty$$

  Not true: $\zeta = 0$ necessitates the full *space-time* scale invariance, while at criticality we have only *spatial* scale-invariance.

What is bulk viscosity at criticality?

- Experiments: typically,

$$\frac{\zeta}{\eta} \lesssim 1$$

  however, for ^{3}He in the vicinity of liquid-vapor critical point

$$\frac{\zeta}{\eta} \gtrsim 10^6$$

- Phenomenology: QCD first order confinement/deconfinement curve (in (T, μ) plane) ends at a critical point of the 3d Ising model universality class. Son-Stephanov (hep-ph/0401052) argued that the dynamical universality class of QCD is that of the liquid-vapor point. For the liquid-vapor critical point Onuki computed:

$$z \approx 3$$

Some theoretical models

- KKT model (A):

$$\zeta_{singular} \propto c_v \propto |t|^{-\alpha}$$

- Quasi-particle models (B):

$$\zeta_{singular} \propto |t|^{\alpha+4\beta-1}$$

- Onuki's dynamical model (C):

$$\zeta_{singular} \propto \xi^{z-\alpha/\nu} \propto |t|^{-z\nu+\alpha}$$

where

$$t = 1 - \frac{T}{T_c}$$

$\Longrightarrow$ above scalings are independent of the number of spatial dimensions

$\Longrightarrow$ vastly different results!!

$\Longrightarrow$ holography to the rescue

$\Longrightarrow$ Using results from $\mathcal{N} = 2^*$ plasma for the critical point with $m_f = 0$:

■ Model A is inconsistent with holographic analysis as it predicts divergent bulk viscosity, $\zeta \ \propto \ |t|^{-1/2}$;
■ Model B does not contradict our holographic analysis as it predicts that $\zeta_{singular} \ \propto \ |t|^{3/2}$;
■ Model C agrees with holographic analysis, provided the dynamical exponent z is

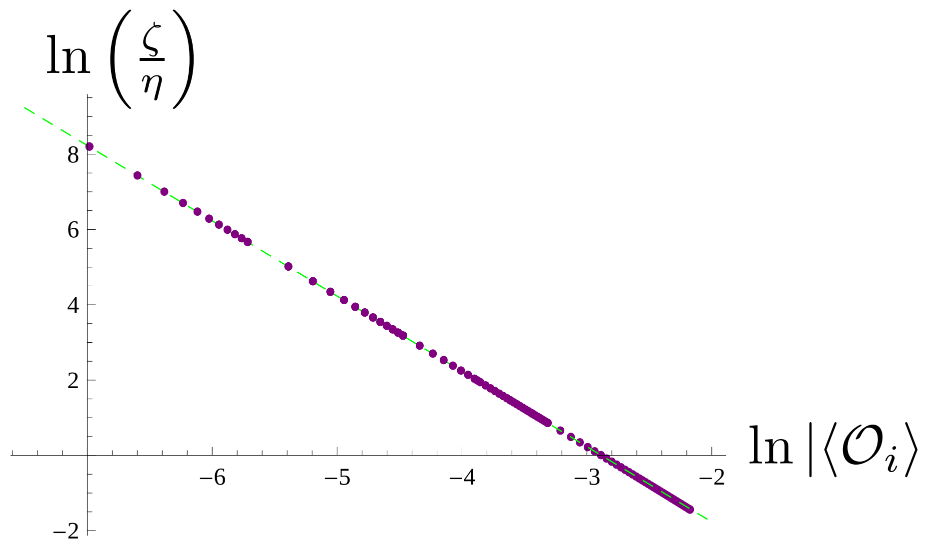
$$z = 1$$

$\Longrightarrow$ Identical results apply to cascading gauge theory

Note: dynamical critical exponent z relates a relation between the divergence of the correlation length and the relaxation time close to criticality:

$$\tau \ \propto \ \xi^z \ \propto \ t^{-\nu z}$$

$\Longrightarrow$ There exist a model where bulk viscosity is divergent at criticality



The ratio of bulk-to-shear viscosities in gauge theory plasma dual to the exotic RG flow in the vicinity of the second order phase transition as a function of the order parameter. The dashed green line represents the linear fit to the log-log data plot with a slope of $(-1.9999(6))$.

$\Longrightarrow$

$$\frac{\zeta}{\eta}\bigg|_{disordered} \propto \ |\langle\mathcal{O}_i\rangle|^{-2} \ \propto |t|^{-1}$$

We conclude:

- once again, Model A is inconsistent with our holographic analysis as it predicts a finite bulk viscosity at the transition, $\zeta \propto |t|^0$;
- Model B is inconsistent as well, as it predicts $\zeta_{singular} \propto |t|^1$;
- Model C agrees with holographic analysis provided the dynamical exponent z is

$$z = 1$$

For the mass deformed $\mathcal{N} = 4$ SYM plasma (with a $U(1)_R$ chemical potential) we find that the bulk viscosity is finite at the critical point with

$$\frac{\zeta}{\eta} = 3.0488(5)\ \left(\frac{1}{3} - c_s^2\right) + \mathcal{O}\left(\left(\frac{1}{3} - c_s^2\right)^2\right)$$

- it satisfies the bulk viscosity bound in strongly coupled plasma

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - c_s^2\right)$$

- with regard to a critical behavior:

$$\frac{\zeta}{\eta} \propto |t|^0$$

We conclude:

- Model A is inconsistent with holographic analysis as it predicts divergent bulk viscosity, $\zeta \ \propto \ |t|^{-1/2}$;
- Model B does not contradict our holographic analysis as it predicts that $\zeta_{singular} \ \propto \ |t|^{3/2}$;
- Model C is inconsistent with holographic analysis as it predicts divergent bulk viscosity, $\zeta \ \propto \ |t|^{-1/2}$;

Actually:

Model B is not applicable as the relaxation time is divergent (model B assumes finite relaxation time):

$$\tau \propto \xi^4 \to \infty \qquad \text{at the transition}$$

Holography eliminated all models of bulk viscosity at criticality!