Holographic thermodynamics

(lecture 1)

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Outline:

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- Gauge/gravity correspondence
- $\mathcal{N} = 2^*$ thermodynamics at strong coupling:
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Basic thermodynamics

$\Longrightarrow$ Consider a stat-mech system with a Hamiltonian H, at equilibrium at temperature T. We can form as partition function as

$$Z = \mathrm{Tr} e^{-\beta \mathrm{H}} = \sum_{\mathrm{n}} \mathrm{e}^{-\beta \mathrm{E_n}}, \qquad \beta \equiv \frac{1}{\mathrm{T}}$$

We can introduce basic thermodynamic potentials at equilibrium (at constant volume):

- the free energy F:

$$Z \equiv e^{-\beta F} \qquad \Longrightarrow \qquad F = -T \ln Z$$

- the energy E:

$$E = Z^{-1} \ \mathrm{Tr}\ \mathrm{H} e^{-\beta \mathrm{H}} = \mathrm{Z}^{-1} \sum_{\mathrm{n}} \mathrm{E_n} \mathrm{e}^{-\beta \mathrm{E_n}} = -\frac{\partial}{\partial \beta} \ln \mathrm{Z}$$

$$= F + \frac{\partial}{\partial \ln \beta} F = F - \frac{\partial}{\partial \ln T} F = F - T \frac{\partial}{\partial T} F$$

- We can also introduce the entropy S as follows. Note that

$$P_n \equiv \frac{1}{Z}\, e^{-\beta E_n}\,, \qquad \sum_n P_n = 1$$

is the probability of a system to occupy a microstate n. The entropy is then

$$S = -\sum_n P_n \,\ln P_n = \frac{1}{Z}\sum_n e^{-\beta E_n}\,(\ln Z + \beta E_n) = \ln Z + \beta E = \beta(E - F)$$

resulting in the basic thermodynamic relation

$$F = E - TS$$

$\Longrightarrow$ It is a simple exercise, given above to establish the **first law** of thermodynamics (at fixed volume)

$$dE = TdS\,, \qquad dF = -SdT$$

$\Longrightarrow$ I emphasize that given the (well-defined) partition function of the system, the basic thermodynamic relation and the first law is **automatic**

$\Longrightarrow$ A comment on QFT:

- Let

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi; \psi, \partial_\mu \psi)$$

be a Lagrangian density of a QFT with bosonic fields ϕ and fermionic fields ψ. Minkowski action is given by

$$S_M = \int d^4x \mathcal{L}$$

and the standard path-integral:

$$\int [d\phi] d[\psi] \; e^{iS_M}$$

- A partition function of the theory evaluate at temperature T is given by the following path-integral

$$Z = \int \; [d\phi] d[\psi] \; e^{-S_E} \; , \qquad S_E = \int dt_E d\vec{x} \mathcal{L}_E$$

where the Euclidean Lagrangian density/action is obtained by the analytical continuation

$$t \to -it_E$$

- The Euclidean time-direction with further compactified with the period $\beta = \frac{1}{T}$:

$$t_E \ \sim \ t_E + \beta$$

with the boundary conditions on the fields:

$$\phi(t_E + \beta) = \phi(t_E)\,, \qquad \psi(t_E + \beta) = -\psi(t_E)$$

<u>CFT and $\mathcal{N} = 2^*$ thermodynamics at weak coupling</u>

$\Longrightarrow$ Start with $\mathcal{N} = 4$ $SU(N)$ SYM. In $\mathcal{N} = 1$ language, the $\mathcal{N} = 4$ multiplet contains a vector multiplet V, an adjoint chiral superfield Φ (related by $\mathcal{N} = 2$ SUSY to V) and an adjoint pair $\{Q, \tilde{Q}\}$ of chiral multiplets, forming an $\mathcal{N} = 2$ hypermultiplet. The theory has a superpotential:

$$W_{CFT} = \frac{2\sqrt{2}}{g_{YM}^2} \mathrm{Tr}\left(\left[Q, \tilde{Q}\right] \Phi\right)$$

The superscript $_{CFT}$ indicates that the theory is classically conformal.

$\Longrightarrow$ To understand the thermodynamics of the theory at weak ($g_{YM} \to 0$) coupling we have to further recall the particle content of $\mathcal{N} = 1$ multiples.

- V contains $SU(N)$ gauge potential A_μ and a superpartner — an adjoint Weyl fermion λ. Thus altogether we have

$$\#\text{bosonic fields}: \qquad 2 \times (\mathrm{N}^2 - 1)$$

$$\#\text{fermionic fields}: \qquad 2 \times (\mathrm{N}^2 - 1)$$

- Each of the chiral multiples $(\Phi, Q, \tilde{Q})$ contain a complex scalar and a Weyl fermion (all in adjoin), adding

$$\#\text{bosonic fields}: \qquad 3 \times 2 \times (\mathrm{N}^2 - 1)$$

$$\#\text{fermionic fields}: \qquad 3 \times 2 \times (\mathrm{N}^2 - 1)$$

- At zero coupling, all interactions can be neglected and $\mathcal{N} = 4$ at thermal equilibrium of finite temperature T is represented by a mixture of an ideal massless relativistic gases of

$$\#\text{bosonic}: \qquad 4 \times 2 \times (\mathrm{N}^2 - 1)$$

$$\#\text{fermionic}: \qquad 4 \times 2 \times (\mathrm{N}^2 - 1)$$

species

$\Longrightarrow$ A single bosonic/fermionic degree of freedom contribute to the energy as

$$E_{b/f} = \int \frac{V d^3k}{(2\pi)^3} \, \frac{\omega(k)}{e^{\beta\omega(k)} \mp 1}\,, \qquad \omega(k) = k$$

$$E_{b/f} = \frac{V}{8\pi^3} \int_0^\infty 4\pi k^2 dk \, \frac{k}{e^{k/T} \mp 1} = \frac{VT^4}{2\pi^2} \times \begin{cases} \frac{\pi^4}{15}\,, & b \\ \frac{7}{8}\,\frac{\pi^4}{15}\,, & f \end{cases}$$

$\Longrightarrow$ Thus, the $\mathcal{N}=4$ energy density at weak coupling is given by

$$\mathcal{E}^{weak}_{\mathcal{N}=4} = 4 \times 2 \times (N^2-1) \times \frac{E_b + E_f}{V}$$

$$\mathcal{E}^{weak}_{\mathcal{N}=4} = \frac{\pi^2}{2} T^4 (N^2-1)$$

The remaining thermodynamic potentials can be deduced from using basic thermodynamics:

$$\mathcal{F}^{weak}_{\mathcal{N}=4} = -P^{weak}_{\mathcal{N}=4} = -\frac{\pi^2}{6} T^4 (N^2-1)$$

$$s^{weak}_{\mathcal{N}=4} = \frac{2\pi^2}{3} T^3 (N^2-1)$$

for the free energy density (pressure) and the entropy density

$\Longrightarrow$ $\mathcal{N}=2^*$ gauge theory is obtained as a massive deformation of $\mathcal{N}=4$ SYM by giving the same mass m to Q and $\tilde{Q}$ chiral multiplets:

$$W_{\mathcal{N}=2^*}=\frac{2\sqrt{2}}{g_{YM}^2}\mathrm{Tr}\left(\left[Q,\tilde{Q}\right]\Phi\right)+\frac{m}{g_{YM}^2}\left(\mathrm{Tr}Q^2+\mathrm{Tr}\tilde{Q}^2\right)$$

Such massive deformation breaks exactly half of the original supersymmetries.

$\Longrightarrow$ Once again, weakly coupled thermodynamics can be studied from the perspective of massive, relativistic bose and fermi gases. Note that exactly $\frac{1}{2}$ of bosonic and $\frac{1}{2}$ fermionic degrees of freedom get mass m. Without derivation, I quote the result for the free energy density of mass m relativistic particle of spin s, $\mathcal{F}_s$,

$$\mathcal{F}_s=-\frac{(2s+1)m^2T^2}{2\pi^2}\sum_{\ell=1}^{\infty}\frac{\eta^{\ell+1}}{\ell^2}K_2\left(\frac{\ell m}{T}\right)$$

where $\eta=\pm1$ for the bosons/fermions correspondingly, and K_2 is a standard Bessel function.

Thus,

$$\mathcal{F}^{weak}_{\mathcal{N}=2^*} = \frac{1}{2}\mathcal{F}^{weak}_{\mathcal{N}=4} - \frac{2m^2T^2(N^2-1)}{\pi^2}\sum_{\ell=1}^{\infty}\frac{1+(-1)^{\ell+1}}{\ell^2}K_2\left(\frac{\ell m}{T}\right)$$

$\Longrightarrow$ Later, we would be interested in the low temperature, i.e. $\frac{m}{T} \gg 1$, behaviour of the $\mathcal{N}=2^*$ thermodynamics. To get this, we can use the asymptotic expansion of K_2:

$$K_2(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}\,, \qquad x \gg 1$$

and find

$$\mathcal{F}^{weak}_{\mathcal{N}=2^*} = -\frac{\pi^2T^4(N^2-1)}{12}\left(1+\frac{24\sqrt{2}}{\pi^{7/2}}\left(\frac{m}{T}\right)^{3/2}\ e^{-\frac{m}{T}}\right)\,, \qquad \frac{m}{T} \gg 1$$

Note that at $m \sim T$,

$$\mathcal{F}^{weak}_{\mathcal{N}=2^*} \approx 0.6\ \mathcal{F}^{weak}_{\mathcal{N}=4}$$

We contrast this with the holographic strong coupling result

Gauge/gravity correspondence

- Gauge/gravity correspondence states

$$Z_{gauge} = Z_{string}\Big|_{\text{certain boundary conditions}}$$

- String theory contains massless (light) and heavy string modes. In certain circumstances there is a hierarchical separation between the modes; furthermore, the light modes can be classical (quantum corrections are suppressed). For example, in $\mathcal{N}=4$ $SU(N)$ SYM with a coupling constant g_{YM}:
  - quantum corrections are suppressed in 't Hooft limit

$$g_{YM}\to 0\,,\qquad N\to\infty\,,\qquad \lambda\equiv N g_{YM}^2 = const$$

  - Supergravity and sting modes are further separated, provided

$$\lambda\to\infty$$

- When it is consistent to truncate to supergravity modes, and treat the latter classically, Z_{string} is dominated by the SUGRA saddle point:

$$Z_{string}\Big|_{\text{certain boundary conditions}} \approx e^{iS_{SUGRA}}\Big|_{EOM,BC}$$

where S_{SUGRA} is a functional of light modes, satisfying classical EOMs with appropriate BC.

- S_{SUGRA} includes:
  - the 10d background metric is a warped product

$$\mathcal{M}_4 \times R_+ \times \Sigma_5$$

where R_+ is a holographic (radial) coordinate r; Σ_5 is a compact manifold which symmetries encode global and supersymmtries of the gauge theory; and $\mathcal{M}_4$ is a four-dimensional manifold fibered over $R_+ \times \Sigma_5$ in such a way that as $r \to +\infty$ it is identified with the background metric of the dual gauge theory (**note:** this is our first encoding of the boundary conditions). For example, consider $\mathcal{N} = 4$ SYM on $R^{3,1}$: then $\mathcal{M}_4$ is asymptotically $R^{3,1}$, and $\Sigma_5 = S^5$ encoding the $SO(6) \sim SU(4)$ R-symmetry of the SYM.

▪ the various RR, and NSNS forms and axion-dilaton, that depend on R_+ and Σ_5 coordinates.

- Since Σ_5 is compact, if it possible to do KK reduction of the 10d type IIB supergravity and obtain effective 5d gravitational action. Since RR/NSNS forms depended only on Σ_5 coordinates, upon KK reduction they become 5d scalars of the effective action:

$$S_5[g_{\mu\nu};\phi_i] = \int d^4x dr \sqrt{-g_5}\ \mathcal{L}_5\,, \qquad \mathcal{L}_5 = \int d^5\Sigma\ \frac{\sqrt{-g_{10}}}{\sqrt{-g_5}}\ \mathcal{L}_{10}$$

- I emphasize that not any 5d effective gravitational action has a meaning as a rigorous gravitational dual to some gauge theory (as lots of models of holographic superconductor); it is only true for actions obtained in a *consistent KK reduction* (as described above). Consistent KK reduction means that **any** solution of 5d effective action can be uplifted (=realized) as a solution of type IIb SUGRA.

- In my lectures I mostly stick with the 'rigorous' examples of duality.

$\mathcal{N} = 2^*$ thermodynamics at strong coupling

- Effective action, EOM, KK uplift, boundary conditions, sample solutions

The effective action (PW) of the five-dimensional supergravity dual to $\mathcal{N} = 2^*$ gauge theory includes metric $g_{\mu\nu}$, and the scalars α and χ

$$S = \frac{1}{4\pi G_5} \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \left[\tfrac{1}{4} R - 3(\partial\alpha)^2 - (\partial\chi)^2 - \mathcal{P}\right]$$

where the potential

$$\mathcal{P} = \frac{1}{16}\left[\frac{1}{3}\left(\frac{\partial W}{\partial\alpha}\right)^2 + \left(\frac{\partial W}{\partial\chi}\right)^2\right] - \frac{1}{3}W^2$$

is a function of α and χ, and is determined by the superpotential

$$W = -e^{-2\alpha} - \frac{1}{2}e^{4\alpha}\cosh(2\chi)\,.$$

The five-dimensional Newton's constant is

$$G_5 \equiv \frac{G_{10}}{2^5\ \mathrm{vol}_{S^5}} = \frac{4\pi}{N^2}\,.$$

The action yields the Einstein equations

$$R_{\mu\nu} = 12\partial_\mu \alpha \partial_\nu \alpha + 4\partial_\mu \chi \partial_\nu \chi + \frac{4}{3} g_{\mu\nu} \mathcal{P}$$

as well as the equations for the scalars

$$\Box \alpha = \frac{1}{6} \frac{\partial \mathcal{P}}{\partial \alpha}\,, \qquad \Box \chi = \frac{1}{2} \frac{\partial \mathcal{P}}{\partial \chi}\,.$$

$\Longrightarrow$ PW action is a consistent truncation of type IIb SUGRA. Specifically, for any 5d solution, the 5d background:

$$ds_5^2 = g_{\mu\nu} dx^\mu dx^\nu\,, \qquad \text{plus} \qquad \{\alpha, \chi\}$$

is uplifted to a solution of 10d type IIB supergravity:

$$ds_{10}^2 = \Omega^2 ds_5^2 + \Omega^2 \frac{4}{\rho^2} \left[\frac{1}{c} d\theta^2 + \rho^6 \cos^2(\theta) \left(\frac{\sigma_1^2}{cX_2} + \frac{\sigma_2^2 + \sigma_3^2}{X_1} \right) + \sin^2(\theta) \frac{1}{X_2} d\phi^2 \right]$$

$$\Omega^2 = \frac{(cX_1X_2)^{1/4}}{\rho}\,, \qquad X_1 = \cos^2\theta + c(r)\rho^6 \sin^2\theta\,, \qquad X_2 = c\cos^2\theta + \rho^6 \sin^2\theta$$

with $c \equiv \cosh 2\chi$, $\rho = e^\alpha$

Plus dilaton-axion, various 3-form fluxes, various 5-form fluxes. σ_i are the left-invariant one forms parameterizing the metric on S^3.

$\Longrightarrow$ We now consider some simple solutions. First of all, note

$$\mathcal{P}(\alpha,\chi) = \frac{1}{4}\left(e^{12\alpha}\cosh^4(\chi) - e^{12\alpha}\cosh^2(\chi) - 4e^{6\alpha}\cosh^2(\chi) + 2e^{6\alpha} - 1\right)e^{-4\alpha}$$

$$\mathcal{P}(0,0) = -\frac{3}{4}\,, \qquad \frac{\partial\mathcal{P}}{\partial\alpha}(0,0) = 0\,, \qquad \frac{\partial\mathcal{P}}{\partial\chi}(0,0) = 0$$

thus, there are solutions where all the scalars are identically zero. The only nontrivial equation in this case is the Einstein equation:

$$R_{\mu\nu} = -g_{\mu\nu}$$

Recall that maximally symmetric d-dimensional anti-deSitter space-time of radius L satisfies

$$AdS_d: \qquad R_{\mu\nu} = -\frac{d-1}{L^2}g_{\mu\nu}$$

thus above can be AdS_5 of radius $L = 2$.

It is instructive to look at the uplifted 10d metric:

$$ds^2_{10(E)} = \Omega^2 ds^2_5 + \Omega^2 \frac{4}{\rho^2}\left[\frac{1}{c}d\theta^2 + \rho^6 \cos^2(\theta)\left(\frac{\sigma_1^2}{cX_2} + \frac{\sigma_2^2+\sigma_3^2}{X_1}\right) + \sin^2(\theta)\frac{1}{X_2}d\phi^2\right]$$

$$\Omega^2 = \frac{(cX_1X_2)^{1/4}}{\rho}\,, \qquad X_1 = \cos^2\theta + c(r)\rho^6\sin^2\theta\,, \qquad X_2 = c\cos^2\theta + \rho^6\sin^2\theta$$

$\Longrightarrow$ We have $\rho = e^\alpha = 1$ and $c = \cosh 2\chi = 1 \Longrightarrow$

$$X_1 = X_2 = 1\,, \qquad \Omega = 1\,, \qquad ds^2_{10(E)} = d(AdS_5)^2 + 2^2\ d(S^5)^2$$

$\Longrightarrow$ We recover that in the limit of vanishing charges, the solution is $AdS_5 \times S^5$ — a well-known dual to $\mathcal{N} = 4$ SYM. Thus, 'turning' on the scalars leads to a mass-deformation $\mathcal{N} = 4 \to \mathcal{N} = 2^*$

Before we move to scalars, let's look more systematically at the solutions of zero-scalar model:

$$R_{\mu\nu} = -g_{\mu\nu}$$

$\Longrightarrow$ We look for solutions that might be dual to thermal states of the SYM theory. Note that finite temperature breaks 4d Lorentz invariance, but preserves (boundary) spatial homogeneity and isotropy; thus we assume

$$ds_5^2 = -c_1(r)^2 \, dt^2 + c_2(r)^2 (d\vec{x})^2 + c_3(r)^2 dr^2$$

Einstein equations for R_{tt} and R_{xx} lead to

$$c_3 \left(\frac{c_1'}{c_3}\right)' + c_1' \, \left(\ln c_2^3\right)' - c_1 c_3^2 = 0$$

$$c_3 \left(\frac{c_2'}{c_3}\right)' + c_2' \, \left(\ln c_1 c_2^2\right)' - c_2 c_3^2 = 0$$

while the Einstein equation for R_{rr} leads to a first-order constraint

$$\frac{c_3^2}{4} - \frac{1}{2} (\ln c_2)' (\ln c_1 c_2)' = 0\,.$$

The constraint equation arises since there is always a reparametrization of the radial coordinate

$$r \to \hat{r}(r)$$

This reparametrization allows to choose arbitrarily one of the functions c_i. Let's choose

$$c_2 = \frac{r}{2}$$

We then find, imposing the boundary condition (the SYM background is $R^{3,1}$)

$$\lim_{r\to\infty} \frac{c_1}{c_2} = 1$$

that

$$c_1 = \frac{r}{2} \left(1 - \frac{r_0^4}{r^4}\right)^{1/2}, \qquad c_3 = \frac{2}{r} \left(1 - \frac{r_0^4}{r^4}\right)^{-1/2}$$

where r_0 is arbitrary.

$\Longrightarrow$ Let's analyze the physical meaning of r_0. Note that as $r \to r_0 + 0$ metric becomes singular. However, this is coordinate singularity only: indeed with

$$r = r_0 + y^2\,, \qquad \Longrightarrow \qquad ds_5^2 \approx -r_0 y^2 dt^2 + \frac{r_0^2}{4}(d\vec{x})^2 + \frac{4}{r_0}\, dy^2$$

Upon analytical continuation $t \to -it_E$,

$$ds_5^2\Big|_E \approx r_0 y^2 dt_E^2 + \frac{r_0^2}{4}(d\vec{x})^2 + \frac{4}{r_0}\, dy^2 = \frac{4}{r_0}\left(y^2(d\gamma)^2 + dy^2\right) + \frac{r_0^2}{4}(d\vec{x})^2$$

$$\gamma \equiv \frac{r_0 t_E}{2}$$

This metric is nonsingular, provided γ is an angle-variable of periodicity 2π:

$$\gamma \sim \gamma + 2\pi \qquad \Longrightarrow \qquad t_E \sim t_E + \frac{4\pi}{r_0}$$

Recall that on QFT side, periodicity of the Euclidean time direction (which is also visible on the boundary) implies the thermal ensemble with

$$T = \left(\frac{4\pi}{r_0}\right)^{-1}$$

On the metric side, periodicity implies that $r = r_0$ is indeed a coordinate singularity, and the Minkowski metric at $r = r_0$ has a regular Schwarzschild horizon.

Having regular horizon, we can follow Bekenstein and Hawking and identify its entropy density (must be density as spatial coordinates are non-compact)

$$s = \frac{A_{horizon}}{4G_5} = \frac{(r_0/2)^3}{4G_5} = \frac{r_0^3}{N^2}128\pi = \frac{1}{2}\pi^2 T^3 N^2$$

where we used identification of G_5 and T. Assuming the standard thermodynamics relations follows (we show in a bit it is), we can also identify

$$\mathcal{F} = -P = -\frac{\pi^2}{8}T^4 N^2 \,, \qquad \mathcal{E} = \frac{3\pi^2}{8}T^4 N^2$$

$\Longrightarrow$ Following holographic relation to the gauge theory we **must** identify above AdS BH thermodynamics with that of the strongly coupled SYM in the 't Hooft limit. Note

$$\mathcal{F}^{strong}_{\mathcal{N}=4} = \frac{3}{4}\mathcal{F}^{weak}_{\mathcal{N}=4} \,, \qquad \mathcal{E}^{strong}_{\mathcal{N}=4} = \frac{3}{4}\mathcal{E}^{weak}_{\mathcal{N}=4} \,, \qquad s^{strong}_{\mathcal{N}=4} = \frac{3}{4}s^{weak}_{\mathcal{N}=4}$$

Notice that even though we introduced a scale into the theory - the temperature - since it is the only physical scale in the problem, the stress-energy tensor of the theory continues to be traceless

$$T^{\mu}_{\mu} = \mathcal{E} - 3P = 0$$

both at strong and weak coupling. In the last lecture we discuss how finite 't Hooft coupling corrections 'patch' the weak and strong coupling thermodynamics.

$\Longrightarrow$ Now I want to comment on a choice of radial coordinate in solving gravitational equations. Our choice was ad-hoc, and we were lucky to find an exact analytic solution. Such 'luck' does not exist in realistic examples of duality. So how do we introduce a radial coordinate?

The drawbacks of our old choice

$$c_2 \equiv \frac{r}{2}$$

are

- the existence of the constraint (the R_{rr} Einstein EOM) reflecting the fact that the radial coordinate diffeomorphism is not fully fixed;
- the range of r

$$r \in [r_0, +\infty)$$

varied as we changed the temperature.

$\Longrightarrow$ Both problems can be solved with the x-coordinate

$$x \equiv 1 - \frac{c_1}{c_2} = 1 - \sqrt{\frac{-g_{tt}}{g_{xx}}}$$

Note:

- such a coordinate can be introduced only for BH-like backgrounds
- independently of the temperature, the range of x is fixed:

$$x \in (0, 1]$$

with $x \to 0_+$ being the asymptotic boundary, and $x \to 1_-$ being the regular Schwarzschild horizon.

$\Longrightarrow$ So how do we rewrite Einstein equations in x coordinate, and how does the constraint disappear?

- Recall we had

$$c_3 \left(\frac{c_1'}{c_3}\right)' + c_1' \ (\ln c_2^3)' - c_1 c_3^2 = 0$$
$$c_3 \left(\frac{c_2'}{c_3}\right)' + c_2' \ (\ln c_1 c_2^2)' - c_2 c_3^2 = 0$$
$$\frac{c_3^2}{4} - \frac{1}{2}(\ln c_2)'(\ln c_1 c_2)' = 0\,.$$

- Write down

$$c_1(r) = c_2(r)(1 - x(r))$$

- Use a linear combination of the first 2 equations to obtain

$$x'' = ((\ln c_3)' - 4(\ln c_2)')\, x'$$

Note that out of these 2 equations we are left with a single one.

- Denote with $\cdot$ the derivative $\frac{d}{dx}$. Using the differentiation chain rule

$$c_2(r)' = \dot{c}_2 \ \frac{dx}{dr}$$

we can solve the constraint equation for $\left(\frac{dx}{dr}\right)^2$:

$$\left(\frac{dx}{dr}\right)^2 = \frac{c_2^2 c_3^2 (x-1)}{2\dot{c}_2(2\dot{c}_2(x-1)+c_2)}$$

- We can use the chain rule

$$c_2'' = \ddot{c}_2 \ \left(\frac{dx}{dr}\right)^2 + \dot{c}_2 \ x''$$

followed up with the substitution for x'', followed up with the chain rule

$$c_2(r)' = \dot{c}_2 \ \frac{dx}{dr}$$

followed up with the substitution for $\left(\frac{dx}{dr}\right)^2$ in the original Einstein equation for c_2, to completely express that equation in x-coordinate:

$$0 = \ddot{c}_2 + \frac{1}{1-x}\ \dot{c}_2 - \frac{5}{c_2}\ \left(\dot{c}_2\right)^2$$

I emphasize that

■ this procedure can easily be generalized for effective actions containing scalars;

■ it eliminates the constraint equation as it focuses on the 'physical' data: the spatial directions warp factor $c_2(x)$ (note that $c_1 = c_2(1-x)$ and is not independent)

■ we can also compute the $g_{xx}dx^2$ part of the metric:

$$g_{xx} = \frac{c_3^2}{\left(\frac{dx}{dr}\right)^2} = 2\ \frac{d(\ln c_2)}{dx} \left(2\ \frac{d(\ln c_2)}{dx} + \frac{1}{x-1}\right)$$

where the last identity is for the specific case of $\mathcal{N} = 4$ SYM dual.

$\Longrightarrow$ Solving $c_2(x)$ equation, subject to divergence as $x \to 0_+$ we find

$$c_2 = \frac{\mu}{(2x - x^2)^{1/4}}$$

where μ is arbitrary integration constant at this stage.

To relate μ to the temperature, we can either repeat the Euclidean continuation procedure, or note that

$$c_2\Big|_{horizon} = \frac{r_0}{2} = \mu \qquad \Longrightarrow \qquad \mu = \frac{r_0}{2} = 2\pi T$$

Finally, as we will be working with x a lot, to keep intuition with the more canonical AdS coordinate r we note (comparing $c_2(r)$ and $c_2(x)$ close to the boundary)

$$\frac{r}{2} = \frac{2\pi T}{(2x - x^2)^{1/4}} \qquad \Longrightarrow \qquad r \sim x^{-1/4}$$

<u>$\mathcal{N} = 2^*$ geometry at strong coupling</u>

Again, taking the metric ansatz to be

$$ds_5^2 = -c_1(r)^2\ dt^2 + c_2(r)^2(d\vec{x})^2 + c_3(r)^2 dr^2\,, \alpha = \alpha(r)\,,\ \chi = \chi(r)$$

we find the following EOMs:

$$c_3\left(\frac{\alpha'}{c_3}\right)' + \alpha' \ \left(\ln c_1 c_2^3\right)' - \frac{1}{6} c_3^2 \frac{\partial \mathcal{P}}{\partial \alpha} = 0$$

$$c_3\left(\frac{\chi'}{c_3}\right)' + \chi' \ \left(\ln c_1 c_2^3\right)' - \frac{1}{2} c_3^2 \frac{\partial \mathcal{P}}{\partial \chi} = 0$$

$$c_3\left(\frac{c_1'}{c_3}\right)' + c_1' \ \left(\ln c_2^3\right)' + \frac{4}{3} c_1 c_3^2 \mathcal{P} = 0$$

$$c_3\left(\frac{c_2'}{c_3}\right)' + c_2' \ \left(\ln c_1 c_2^2\right)' + \frac{4}{3} c_2 c_3^2 \mathcal{P} = 0$$

plus a constraint:

$$(\alpha')^2 + \frac{1}{3}(\chi')^2 - \frac{1}{3} c_3^2 \mathcal{P} - \frac{1}{2}(\ln c_2)'(\ln c_1 c_2)' = 0$$

Note that the scalar backreact on the geometry quadratically:

$$\mathcal{P} = -\frac{3}{4} + \left(-3\alpha^2 - \frac{3}{4}\chi^2\right) + \cdots$$

They also have negative mass^2!

Typically, a negative mass^2 implies instability — this is true in flat space, but not in AdS. In fact, in AdS$_{(d+1)}$ we have the so-called Breitenlohner-Freedman:

$$L^2 m^2 \geq -\frac{d^2}{4}$$

to avoid the instabilities (assuming the scalar field kinetic term in the action is canonically normalized).

$\Longrightarrow$ Consider the behaviour of the (linearized) scalar field EOM close to the boundary:

$$0 = (\underset{AdS}{\Box} - m^2)\phi$$

with the AdS metric of the form

$$ds_5^2 = \frac{r^2}{L^2}(-dt^2 + (d\vec{x})^2) + \frac{L^2}{r^2}dr^2$$

and assuming that $\phi = \phi(r)$ we find

$$0 = \frac{L^3}{r^3}\left(\frac{r^5}{L^5}\phi'\right)' - m^2\phi$$

As $r \to \infty$, we find

$$\phi \sim \phi_{non-normalizable}\,\frac{1}{r^{4-\Delta}} + \phi_{normalizable}\,\frac{1}{r^{\Delta}}\,, \qquad m^2L^2 = \Delta(\Delta-4)$$

I don't have time to explain this, but according to holographic dictionary, if a CFT is deformed by an operators $\mathcal{O}$ with a coupling constant λ, i.e,

$$\mathcal{L} = \mathcal{L}_{CFT} + \lambda\mathcal{O}$$

then

- $\Delta = dim(\mathcal{O})$
- $\lambda = \phi_{non-normalizable}$
- $\langle\mathcal{O}\rangle = \phi_{normalizable}$

Doing this analysis for the PW effective action we find

- α is a scalar dual to a dim-2 operator $\mathcal{O}_2$
- χ is a scalar dual to a dim-3 operator $\mathcal{O}_3$ This is expected as $\mathcal{N} = 2^*$ is obtained from $\mathcal{N} = 4$ by giving a mass to 2 chiral superfields, i.e, a mass to scalars (a dim-2 operator) and a mass to corresponding Weyl fermions (a dim-3 operator).

It is possible to precisely identify the dual operators:

$$\mathcal{L}_{\mathcal{N}=2^*} = \mathcal{L}_{\mathcal{N}=4} + \delta\mathcal{L}$$

$$\delta\mathcal{L} = -2 \int d^4x \, \left[m^2 \, \mathcal{O}_2 + m \, \mathcal{O}_3 \right]$$

with

$$\mathcal{O}_2 = \frac{1}{3} \mathrm{Tr} \left(\, |\phi_1|^2 + |\phi_2|^2 - 2 \, |\phi_3|^2 \, \right)$$

$$\mathcal{O}_3 = -\mathrm{Tr} \left(i \, \psi_1 \psi_2 - \sqrt{2} g_{YM} \, \phi_3 [\phi_1, \phi_1^\dagger] + \sqrt{2} g_{YM} \, \phi_3 [\phi_2^\dagger, \phi_2] + \text{h.c.} \right)$$
$$+ \frac{2}{3} m \, \mathrm{Tr} \left(\, |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 \, \right)$$

$\Longrightarrow$ There is an exact analytical solution when scalars are turned off (this is just $\mathcal{N} = 4$ SYM dual); there is also an analytical solution at $T = 0$ - the original (supersymmetric) Pilch-Warner solution.

PW geometry ansatz:

$$ds_5^2 = e^{2A}\left(-dt^2 + d\vec{x}^{\,2}\right) + d\hat{x}^2$$

The second order equations of motion for $A\,,\alpha\,,\chi$, because of the supersymmetry, is equivalent to first-order EOMS:

$$\frac{dA}{d\hat{x}} = -\frac{1}{3}W\,, \qquad \frac{d\alpha}{d\hat{x}} = \frac{1}{4}\frac{\partial W}{\partial \alpha}\,, \qquad \frac{d\chi}{d\hat{x}} = \frac{1}{4}\frac{\partial W}{\partial \chi}$$

Solutions to above are characterized by a single parameter k:

$$e^A = \frac{k\rho^2}{\sinh(2\chi)}\,, \qquad \rho^6 = \cosh(2\chi) + \sinh^2(2\chi)\,\ln\frac{\sinh(\chi)}{\cosh(\chi)}$$

I am not going to discuss this, but it is possible (because $\mathcal{N}=2^*$ is solvable even at strong coupling) to match the gravitational parameter k to a gauge theory mass m,

$$k = 2m$$

To get better understanding of the solution, introduce the 'standard' AdS coordinate r as

$$\frac{1}{r} \equiv e^{-\hat{x}/2} \,,$$

then

$$\chi = \frac{k}{r}\biggl[1 + \frac{k^2}{r^2}\left(\tfrac{1}{3} - \tfrac{4}{3}\ln(kr)\right) + \frac{k^4}{r^4}\left(-\tfrac{7}{90} - \tfrac{10}{3}\ln(kr) + \tfrac{20}{9}\ln^2(kr)\right) + \mathcal{O}\left(\frac{k^6}{r^6}\ln^3(kr)\right)\biggr]$$

$$\rho = 1 + \frac{k^2}{r^2}\left(\tfrac{1}{3} - \tfrac{2}{3}\ln(kr)\right) + \frac{k^4}{r^4}\left(\tfrac{1}{18} - 2\ln(kr) + \tfrac{2}{3}\ln^2(kr)\right) + \mathcal{O}\left(\frac{k^6}{r^6}\ln^3(kr)\right)$$

$$A = \ln(r/2) - \tfrac{1}{3}\frac{k^2}{r^2} - \frac{k^4}{r^4}\left(\tfrac{2}{9} - \tfrac{10}{9}\ln(kr) + \tfrac{4}{9}\ln^2(kr)\right) + \mathcal{O}\left(\frac{k^6}{r^6}\ln^3(kr)\right)$$

That is:

$$e^{2A} \propto \frac{r^2}{4}, \qquad \alpha \propto -\frac{2}{3}\frac{k^2\ln r}{r^2}\,, \qquad \chi \propto \frac{k}{r}\,, \qquad r \to \infty$$

$\Longrightarrow$ Notice that the nonnormalizable components of $\{\alpha, \chi\}$ are related — this is holographic dual to $\mathcal{N} = 2$ susy preserving condition on the gauge theory side:

$$m_b = m_f$$

But in general, we can keep $m_b \neq m_f$:

$$e^{2A} \propto \frac{r^2}{4}, \qquad \alpha \propto \frac{m_b^2 \ln r}{r^2}, \qquad \chi \propto \frac{m_f}{r}, \qquad r \to \infty$$

The precise relation, including numerical coefficients can be worked out.

$\Longrightarrow$ There are no singularity-free flows (geometries) with $m_b \neq m_f$ and at zero temperature $T = 0$. However, one can study $m_b \neq m_f$ mass deformations of $\mathcal{N} = 4$ SYM at finite temperature.

To study thermodynamics we introduce the x-coordinate as before, namely

$$x \equiv 1 - \frac{c_1}{c_2}$$

Repeating procedure discussed earlier we find

$$c_2'' + 4c_2\ (\alpha')^2 - \frac{1}{x-1} c_2' - \frac{5}{c_2}(c_2')^2 + \frac{4}{3} c_2\ (\chi')^2 = 0$$

$$\alpha'' + \frac{1}{x-1}\ \alpha' - \frac{\frac{\partial \mathcal{P}}{\partial \alpha}}{12\,\mathcal{P} c_2^2 (x-1)} \Big[(x-1) \left(6(\alpha')^2 + 2(\chi')^2\right) c_2^2 - 3c_2' c_2 - 6(c_2')^2 (x-1) \Big]$$

$$\chi'' + \frac{1}{x-1}\ \chi' - \frac{\frac{\partial \mathcal{P}}{\partial \chi}}{4\,\mathcal{P} c_2^2 (x-1)} \Big[(x-1) \left(6(\alpha')^2 + 2(\chi')^2\right) c_2^2 - 3c_2' c_2 - 6(c_2')^2 (x-1) \Big] =$$

We look for a solution to above subject to the following (fixed) boundary conditions:

$\Longrightarrow$ near the boundary, $x \propto r^{-4} \to 0_+$

$$\left\{c_2(x), \alpha(x), \chi(x)\right\} \to \left\{x^{-1/4}, \qquad \frac{m_b^2}{T^2}\ x^{1/2} \ln x, \qquad \frac{m_f}{T}\ x^{1/4}\right\}$$

of course, we need a precise coefficients here relating the non-normalizable components of the sugra scalars to the gauge theory masses

$\Longrightarrow$ near the horizon, $x \to 1_-$ (to have a regular, non-singular Schwarzschild horizon)

$$\left\{c_2(x), \alpha(x), \chi(x)\right\} \to \left\{\text{constant}\,, \qquad \text{constant}\,, \qquad \text{constant}\right\}$$

$\Longrightarrow$ I will discuss now in details the case when the scalar amplitudes are small, as we will see, equivalent to the high-T thermodynamics.

Introduce

$$c_2(x) = e^{A(x)}$$

We will look for the solutions to quadratic order in δ_1, δ_2:

$$A(x) = \ln \delta_3 - \frac{1}{4}\ \ln\left(2x - x^2\right) + \delta_1^2\ A_1(x) + \delta_2^2\ A_2(x) + \mathcal{O}(\delta_{1,2}^4)$$

$$\alpha(x) = \delta_1\ \alpha_1(x) + \mathcal{O}(\delta_{1,2}^3)\,, \qquad \chi(x) = \delta_2\ \chi_2(x) + \mathcal{O}(\delta_{1,2}^3)$$

where to fix the normalization of δ_i we assume $\alpha_1(x=1) = \chi_2(x=1) = 1$, and $A_i(x=1) = 0$.

Note that with $\delta_1 = \delta_2 = 0$ we have AdS BH solution with

$$2\pi T = \delta_3$$

to quadratic order (again, doing Euclidean analysis)

$$2\pi T = \delta_3 \left(1 + \delta_1^2 \left(2 - \frac{d^2 A_1}{dx^2}\bigg|_{x=1}\right) + \delta_2^2 \left(\frac{1}{2} - \frac{d^2 A_2}{dx^2}\bigg|_{x=1}\right)\right)$$

Also, from the horizon area density we can compute the entropy density as

$$s = \frac{e^{3A(x=1)}}{4G_5} = \frac{\delta_3^3}{4G_5}\left(1 + 3\delta_1^2 A_1\bigg|_{x=1} + 3\delta_2^2 A_2\bigg|_{x=1} + \mathcal{O}(\delta_{1,2}^4)\right)$$

To order $\mathcal{O}(\delta_{1,2})$ (I now use primes to denote derivatives $\frac{d}{dx}$),

$$0 = \alpha_1'' + \frac{1}{x-1}\ \alpha_1' + \frac{1}{(x-2)^2x^2}\ \alpha_1$$

$$0 = \chi_2'' + \frac{1}{x-1}\ \chi_2' + \frac{3}{4(x-2)^2x^2}\ \chi_2$$

Which can be solved, subject to the boundary conditions prescribed:

$$\alpha_1 = (2x-x^2)^{1/2}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;(1-x)^2\right)$$

$$\chi_2 = (2x-x^2)^{3/4}\ {}_2F_1\left(\frac{3}{4},\frac{3}{4};1;(1-x)^2\right)$$

To order $\mathcal{O}(\delta_{1,2}^2)$ we find

$$0 = A_1'' + \frac{3x^2-6x+4}{x(x-1)(x-2)}A_1' + 4(\alpha_1')^2$$

$$0 = A_2'' + \frac{3x^2-6x+4}{x(x-1)(x-2)}A_2' + \frac{4}{3}(\chi_2')^2$$

The latter equations can be solved as well:

$$A_1 = 4\int_x^1 \frac{(z-1)dz}{(2z-z^2)^2}\left(\gamma_1 - \int_z^1 dy\left(\frac{\partial\alpha_1}{\partial y}\right)^2\frac{(2y-y^2)^2}{y-1}\right)$$

$$A_2 = \frac{4}{3}\int_x^1 \frac{(z-1)dz}{(2z-z^2)^2}\left(\gamma_2 - \int_z^1 dy\left(\frac{\partial\chi_2}{\partial y}\right)^2\frac{(2y-y^2)^2}{y-1}\right)$$

The constants γ_i are fine-tuned to satisfy the boundary conditions, and are given by

$$\gamma_1 = \frac{8-\pi^2}{2\pi^2}\,, \qquad \gamma_2 = \frac{8-3\pi}{8\pi}\,.$$

$\Longrightarrow$ We are almost done: what is left is to relate δ_1 and δ_2 to m_b and m_f. To do this, we match with the asymptotic PW solution. Indeed, using

$$\alpha \approx -\frac{2}{3}\frac{k^2\ln r}{r^2} = -\frac{2}{3}\frac{(2m_b)^2\ln r}{r^2}\,, \qquad \chi \approx \frac{k}{r} = \frac{2m_f}{r}\,, \qquad e^A = \frac{r}{2}$$

and

$$\alpha = \delta_1\alpha_1(x\to 0) \approx -\frac{\delta_1(2x)^{1/2}\ln x}{\pi}\,, \qquad \chi = \delta_2\chi_2(x\to 0) \approx \delta_2\frac{(2x)^{1/4}\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)^2}$$

$$e^A \approx \delta_3(2x)^{-1/4} = 2\pi T(2x)^{-1/4}$$

We can match e^A to identify

$$r \approx \frac{4\pi T}{(2x)^{1/4}}$$

leading to

$$\delta_1 = -\frac{1}{24\pi} \left(\frac{m_b}{T}\right)^2 , \qquad \delta_2 = \frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^2}{2\pi^{3/2}} \frac{m_f}{T}$$

We can now compute the entropy density

$$s = \frac{1}{2}\pi^2 N^2 T^3 \left(1 - \frac{48}{\pi^2} \delta_1^2 - \frac{4}{\pi} \delta_2^2\right)$$

and using the first law of thermodynamics deduce

$$\mathcal{F} = -\frac{1}{8}\pi^2 N^2 T^4 \left[1 - \frac{192}{\pi^2} \ln(\pi T) \delta_1^2 - \frac{8}{\pi} \delta_2^2\right]$$

$\Longrightarrow$ I don't have time to discuss this, but it is possible to use the technology of the holographic renormalization to compute $\mathcal{F}$ and $\mathcal{E}$ independently, and explicitly verify that the first law of thermodynamics is valid.

$\Longrightarrow$ How do we do thermodynamics in general?

Introduce

$$A \equiv \ln \delta_3 - \frac{1}{4} \ln(2x - x^2) + a(x)$$

- We begin and solving the full nonlinear system of 3 second-order ODEs for $\{a, \alpha, \chi\}$ perturbatively near the boundary (as $x \to 0$) and near the horizon (as $y \equiv (1-x) \to 0$). We find:

$$\rho = e^{\alpha} = 1 + x^{1/2} \left(\rho_{10} + \rho_{11} \ \ln x\right) + \dots$$

$$\chi = \ \chi_0 x^{1/4} \left[1 + x^{1/2} \left(\chi_{10} + \frac{1}{3}\chi_0^2 \ \ln x\right) + \cdots\right]$$

$$a = \ x^{1/2} \left(-\frac{1}{9}\chi_0^2\right) + \dots$$

$$\rho = e^{\alpha} = \rho_0^h + \mathcal{O}(y^2) \,, \qquad \chi = \chi_0^h + \mathcal{O}(y^2) \,, \qquad a = a_0^h + a_1^h y^2 + \cdots$$

where the higher order terms denoted by $\dots$ are completely determined by explicitly specified parameters. Altogether the 9 independent parameters are:

$$\{ \ \delta_3, \rho_{10}, \rho_{11}, \chi_0, \chi_{10}, \rho_0^h, \chi_0^h, a_0^h, a_1^h\}$$

- The 3 combinations of the parameters are related to $\{T, m_b, m_f\}$ as

$$T = \frac{\delta_3}{2\pi}\ e^{-3a_0^h}\,, \qquad \rho_{11} = \frac{\sqrt{2}}{24\pi^2}\ e^{-6a_0^h}\ \left(\frac{m_b}{T}\right)^2$$

$$\chi_0 = \frac{1}{2^{3/4}\pi}\ e^{-3a_0^h}\ \left(\frac{m_f}{T}\right)$$

- The remaining 6 are functions of the above, and completely (uniquely) specify the solution for the system $\{a, \alpha, \chi\}$: indeed, 3 second order equations need 6 parameters to determined solution. Note that δ_3 is an overall scaling parameter, and does not enter into determining the numerical solution — it is used though to translate the holographic observables to gauge theory observables. Basically, $T \propto \delta_3$ and solving the system for $\{a, \alpha, \chi\}$ determines the scaled thermodynamics variables

$$\frac{\mathcal{F}}{T^4}\,, \qquad \frac{\mathcal{E}}{T^4}\,, \qquad \frac{s}{T^3}$$

as a function of

$$\frac{m_f}{T}\,, \qquad \frac{m_b^2}{T^2}$$

- The numerical procedure is a 'shooting method':
  - fix $\{\rho_{11}, \chi_0\}$
  - start with a 'seed' values for

$$\{\rho_{10}, \chi_{10}, \rho_0^h, \chi_0^h, a_0^h, a_1^h\}$$

  - numerically integrate $\{a, \alpha, c\}$ from the boundary and the horizon using the asymptotic expansion described above from $x \in [0, 1/2]$ and from $y \in [0, 1/2]$
  - construct a 'mismatch vector' at $x = y = \frac{1}{2}$:

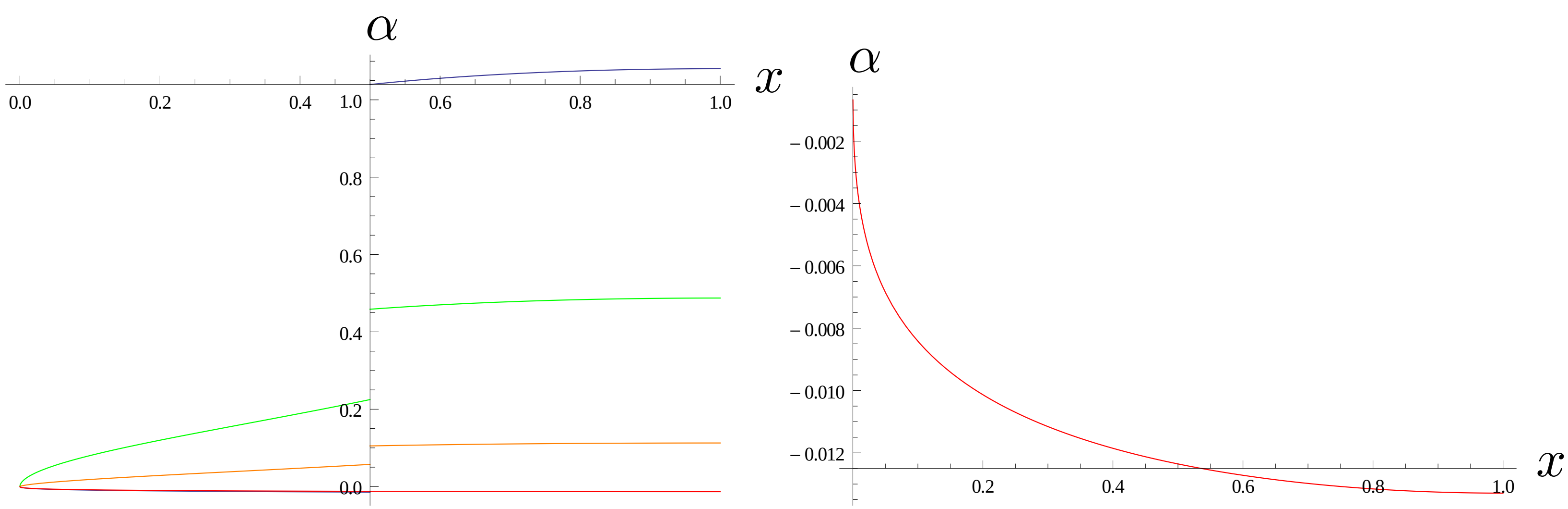
$$\vec{v} = \left(a^b - a^h, (a^b)' + (a^h)', \alpha^b - \alpha^h, (\alpha^b)' + (\alpha^h)', \chi^b - \chi^h, (\chi^b)' + (\chi^h)' \right)$$

  - note that for exact solution,

$$||\vec{v}|| = 0$$

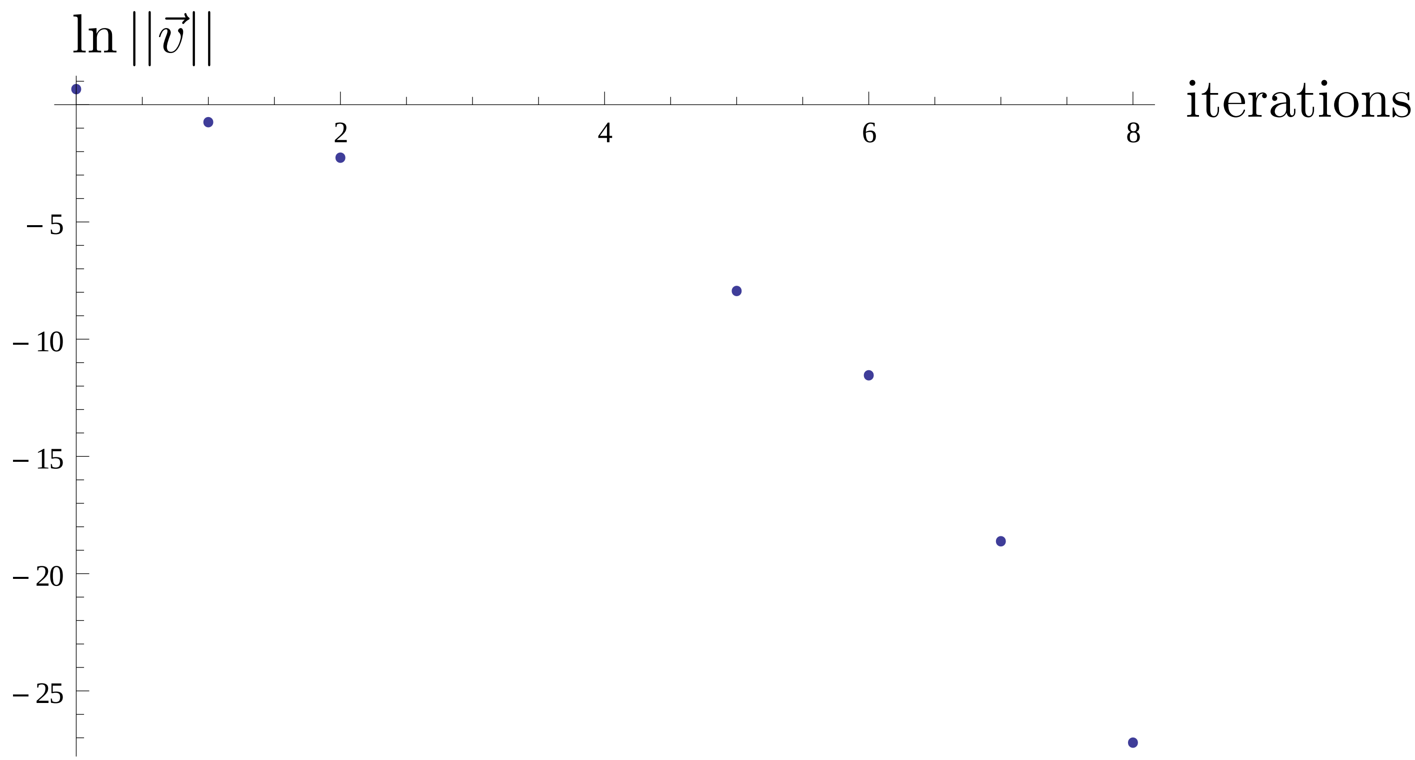
  - in practice, $||\vec{v}|| \neq 0$, and we use the steepest decent in adjusting $\{\rho_{10}, \chi_{10}, \rho_0^h, \chi_0^h, a_0^h, a_1^h\}$ to decrease the norm of $\vec{v}$.

$\Longrightarrow$ It is straightforward to construct an efficient numerical algorithm for the steepest descent:



Convergence of the solution for $0, 1, 2, 5$ iterations (blue,green,orange,red)

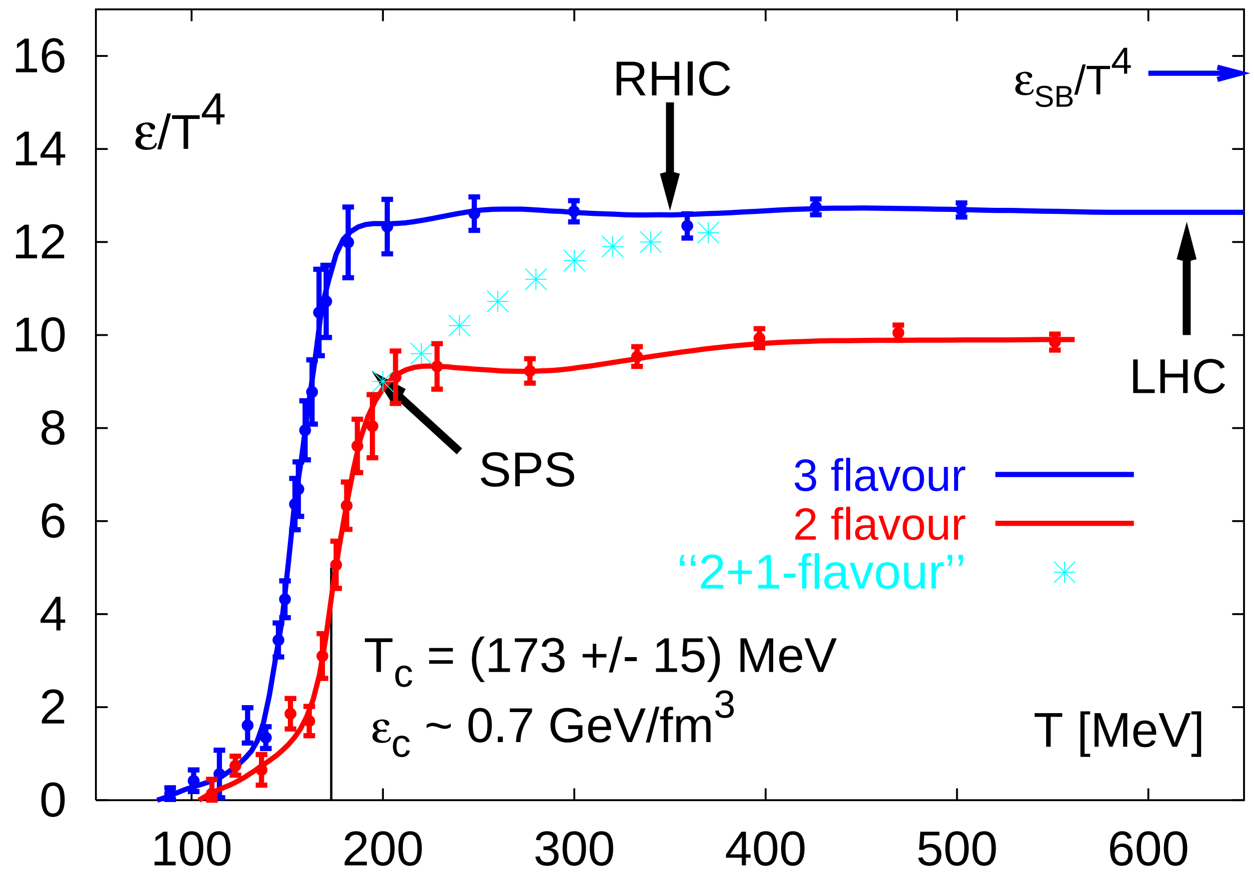
$\Longrightarrow$ The convergence of the mismatch vector norm:



After 8 iterations the norm is $\sim 10^{-12}$ (for seed the norm is 1.9)

$\Longrightarrow$ Once we accumulate gravity data, we can translate it into gauge theory variables.

Before we discuss the $\mathcal{N} = 2^*$ thermodynamics, recall the lattice data for the QCD:



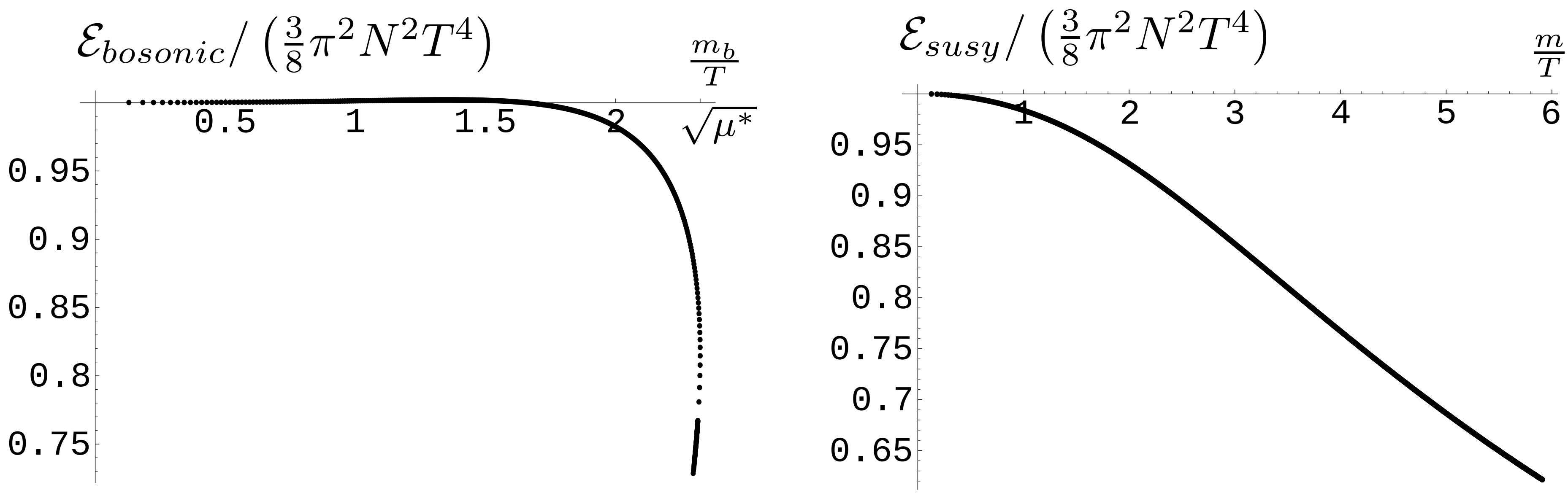
QCD thermodynamics from lattice; F.Karsch and E.Laermann, hep-lat/0305025.

- RHIC QGP is strongly coupled because equilibrium plasma temperature is roughly the QCD deconfinement temperature,

$$T_{plasma} \sim T_{deconfinement} \sim \Lambda_{QCD}$$

- Thus scale invariance is strongly broken and it is not clear why conformal $\mathcal{N} = 4$ plasma or near-conformal plasma thermodynamics/hydrodynamics should be relevant...

Surprisingly...



Equation of state of the mass deformed $\mathcal{N}=4$ gauge theory plasma. At $T \sim m$ the deviation from the conformal thermodynamics is $\sim 2\%$. Recall from the early slides, for the ideal gas approximation the deviation is about 40%.

$\Longrightarrow \mathcal{N}=2^*$ model appears to share a 'thermodynamic plateau' with QCD!

$\Longrightarrow$ However, $\mathcal{N} = 2^*$ is not a QCD. A notable different is that this model does not have a confinement/deconfinement phase transition.

- In the deconfined phase the free energy density and the entropy density

$$\mathcal{F}_{deconfined} \propto \mathcal{O}\left(N^2\right) , \qquad s_{deconfined} \propto \mathcal{O}\left(N^2\right)$$

- In the confined phase the free energy density

$$\mathcal{F}_{confined} \propto \mathcal{O}\left(N^0\right) , \qquad s_{confined} \propto \mathcal{O}\left(N^0\right)$$

- Since

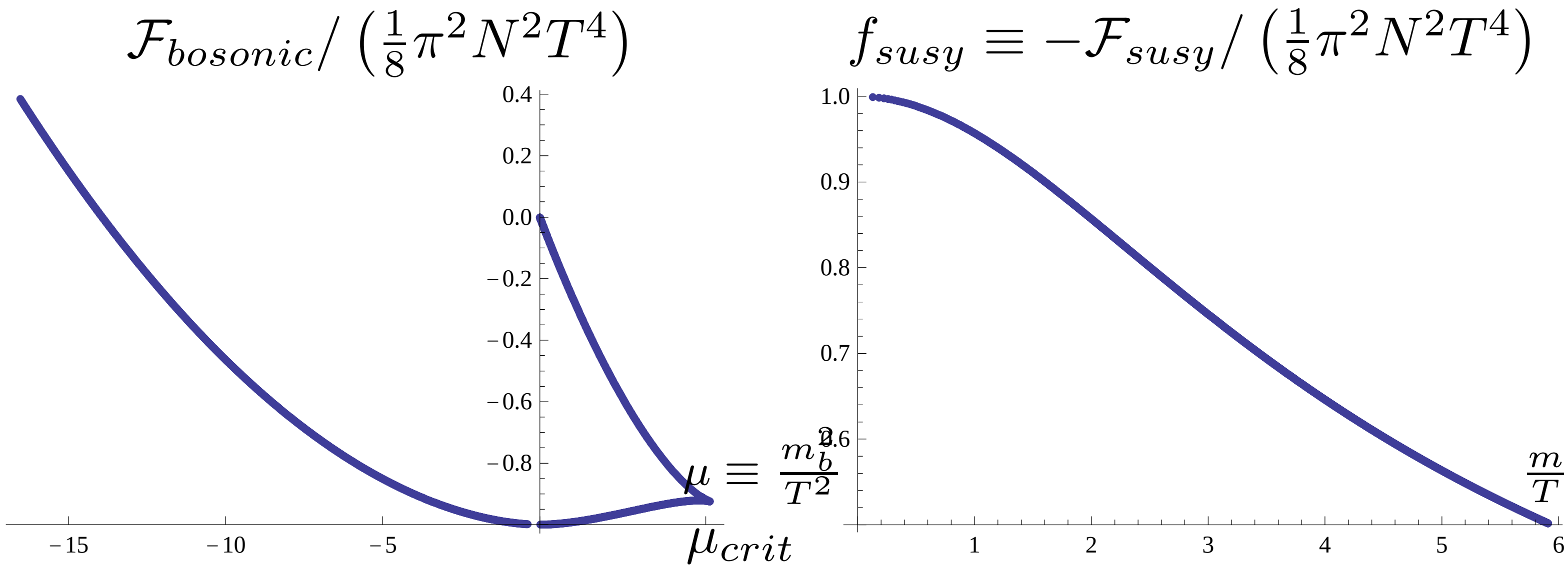
$$\lim_{N\to\infty} \left.\frac{\mathcal{F}}{N^2}\right|_{deconfined} \neq 0 \,, \qquad \text{or} \qquad \lim_{N\to\infty} \left.\frac{\mathcal{F}}{sT}\right|_{deconfined} \neq 0$$

  and

$$\lim_{N\to\infty} \left.\frac{\mathcal{F}}{N^2}\right|_{confined} = 0 \,,$$

  the confined phase of plasma is thermodynamically favourable once

$$\frac{\mathcal{F}}{sT} > 0 \,, \qquad \text{provided} \qquad s \sim \mathcal{O}\left(N^2\right)$$



The left plot represents bosonic mass deformation free energy $\mathcal{F}_{bosonic}$ as a function of $\mu \equiv \frac{m_b^2}{T^2}$. The left plot represents supersymmetric mass deformation free energy $\mathcal{F}_{susy}$ as a function of $\frac{m}{T}$.

$\Longrightarrow$ Note that in the physical domain, i.e, $m_{b,f}^2 > 0$,

$$\mathcal{F} < 0$$

Thus, as claimed, there is no confinement/deconfinement transition in this model.

Critical phenomena in $\mathcal{N} = 2^*$

The phase diagram of the $\mathcal{N} = 2^*$ model depends on

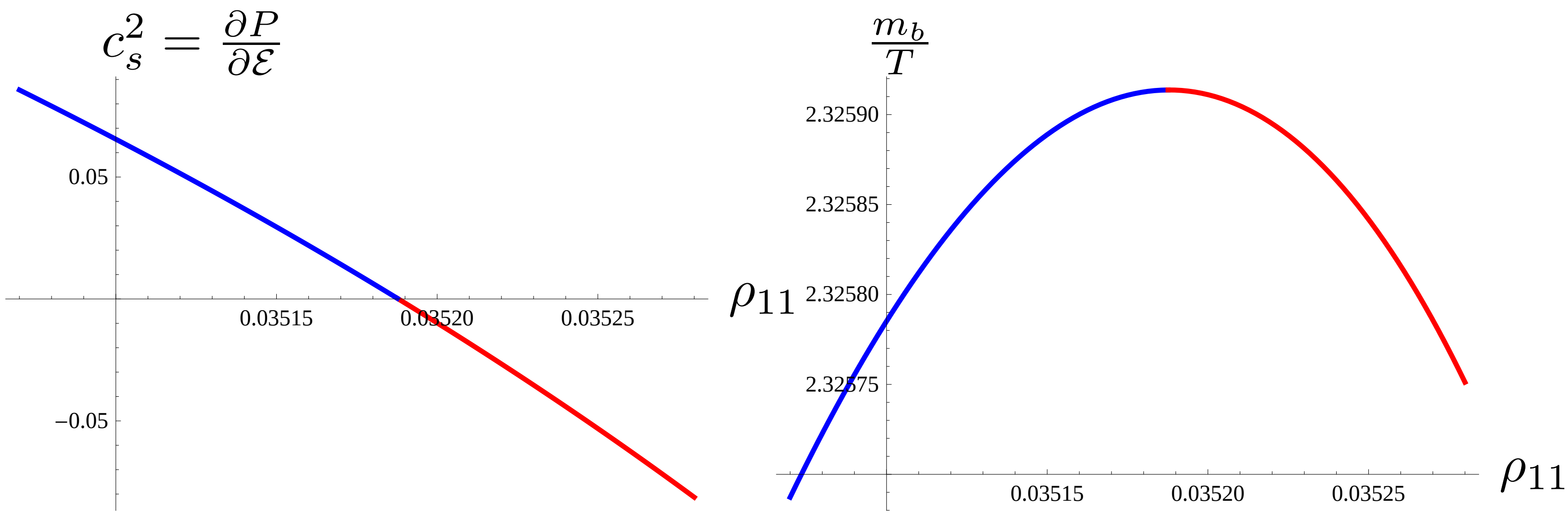
$$\Delta \equiv \frac{m_f^2}{m_b^2} :$$

- when $\Delta \ge 1$ there is no phase transition in the system;
- when $\Delta < 1$ there is a critical point in the system with the divergent specific heat. The corresponding critical exponent is $\alpha = 0.5$:

  $$c_V \sim |1 - T_c/T|^{-\alpha}$$

  where $T_c = T_c(\Delta)$.

$\Longrightarrow$ Let's focus on $m_f = 0$, $m_b \neq 0$

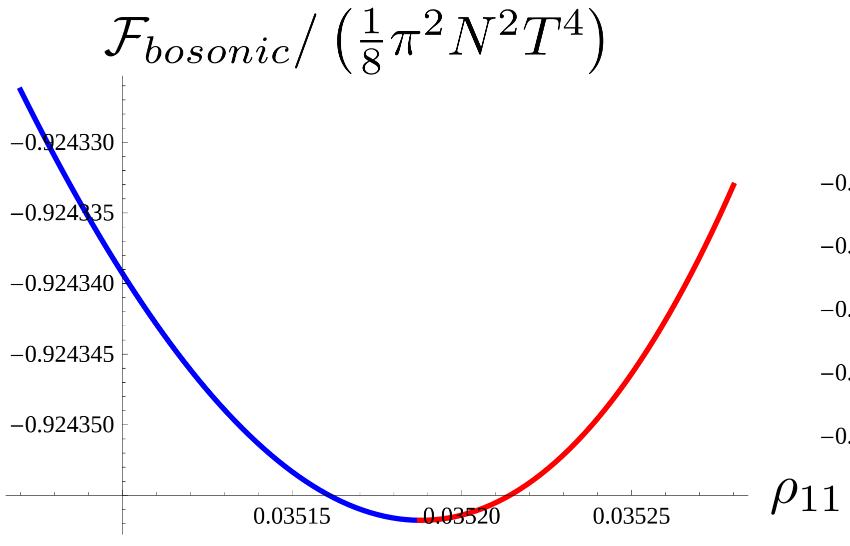


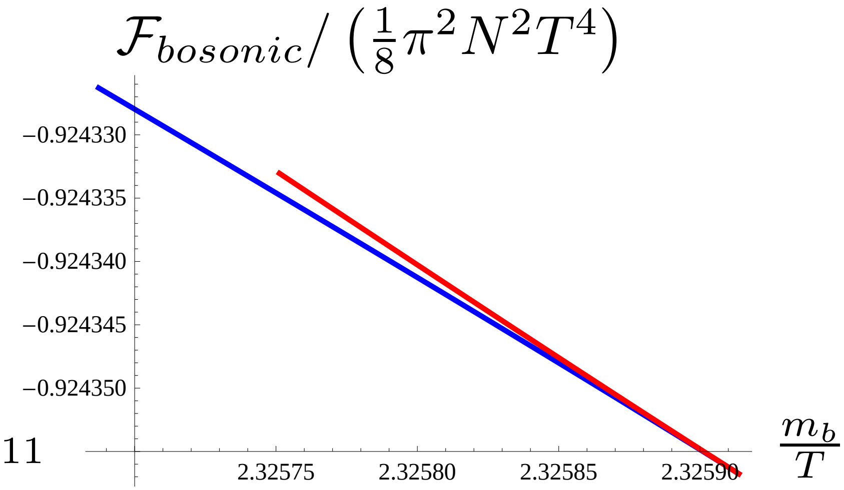
The speed of sound c_s^2 (left plot) and the reduced temperature $\frac{m_b}{T}$ (right plot) of the strongly coupled $\mathcal{N} = 2^*$ plasma with $m_f = 0$ and $m_b \neq 0$ as a function of the dual gravitation parameter ρ_{11}.

Introduce

$$\Delta\rho_{11} = \rho_{11} - \rho_{11}^c$$

$$t \equiv 1 - \frac{T_c}{T} \ \propto \ (\Delta\rho_{11})^2 \,, \qquad c_s^2\Big|_{blue} \ \propto \ \left(-c_s^2\right)\Big|_{red} \ \propto \ |\Delta\rho_{11}| \ \propto \ t^{1/2}$$





Free energy densities $\mathcal{F}_{boson}$ as a function of ρ_{11} (left plot) and $\frac{m_b}{T}$ (right plot) of the $\mathcal{N}=2^*$ plasma with $m_f=0$.

$$c_V = -T\left(\frac{\partial^2\mathcal{F}}{\partial T^2}\right) = \frac{s}{c_s^2} \propto t^{-1/2} \qquad \Longrightarrow \qquad \alpha = \frac{1}{2}$$

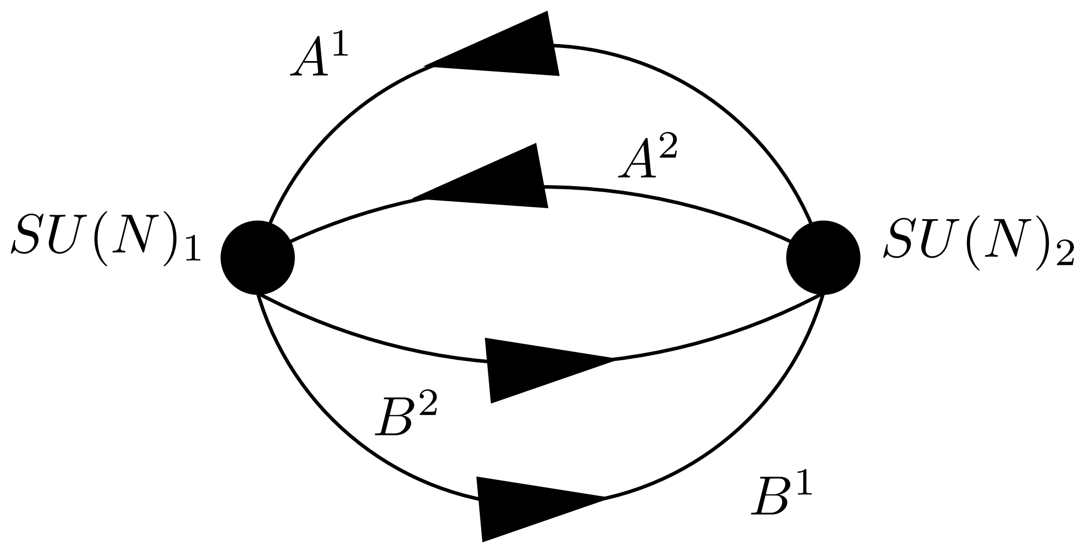
$\Longrightarrow$ Notice that there is a relation between the negative specific heat ($c_V < 0$ - a thermodynamics instability), and the instability of the sound modes ($c_s^2 < 0$ - a dynamical instability)

Klebanov-Strassler model (a QFT story)

$\Rightarrow$ The staring point again is $\mathcal{N}=4$ SU(N) SYM.

- Consider a $\mathbb{Z}_2$ orbifold of above SYM:

$$\mathcal{N}=4 \qquad \rightarrow \qquad \mathcal{N}=2$$



	$SU(N)_1 \times SU(N)_2$
A^i :	$(\bar{N}, N)$
B^i :	$(N, \bar{N})$

$$\mathcal{W}_{\mathcal{N}=2} = g_1 \;\operatorname{Tr}\Phi_1\left[A^1B^1 + A^2B^2\right] + g_2\;\operatorname{Tr}\Phi_2\left[B^1A^1 + B^2A^2\right]$$

Note: $\beta_i = 0 \Longrightarrow g_1, g_2$ are exactly marginal couplings

- Turn on the mass term that breaks SUSY $\mathcal{N}=2 \ \rightarrow \ \mathcal{N}=1$

$$\mathcal{W}_{\mathcal{N}=2} \rightarrow \mathcal{W}_{\mathcal{N}=1} = \mathcal{W}_{\mathcal{N}=2} + m \ \operatorname{Tr}\left(\Phi_1^2 - \Phi_2^2\right)$$

$\Longrightarrow$ Integrating out the massive fields we find

$$\mathcal{W}_{eff} = \lambda \operatorname{Tr} A^i B^j A^k B^\ell \epsilon^{ik} \epsilon^{j\ell}$$

$\Longrightarrow$ Klebanov and Witten argued that at energy scales $\ll m$ the theory flows to a strongly interactive superconformal field theory; the coupling λ is exactly marginal, and thus the fields A^i, B^j develop large anomalous dimensions

$$[A^i]^{UV} = 1 \quad \rightarrow \quad [A^i]^{IR} = \frac{3}{4} \qquad \Longrightarrow \ \gamma_{A^i} = -\frac{1}{4}$$

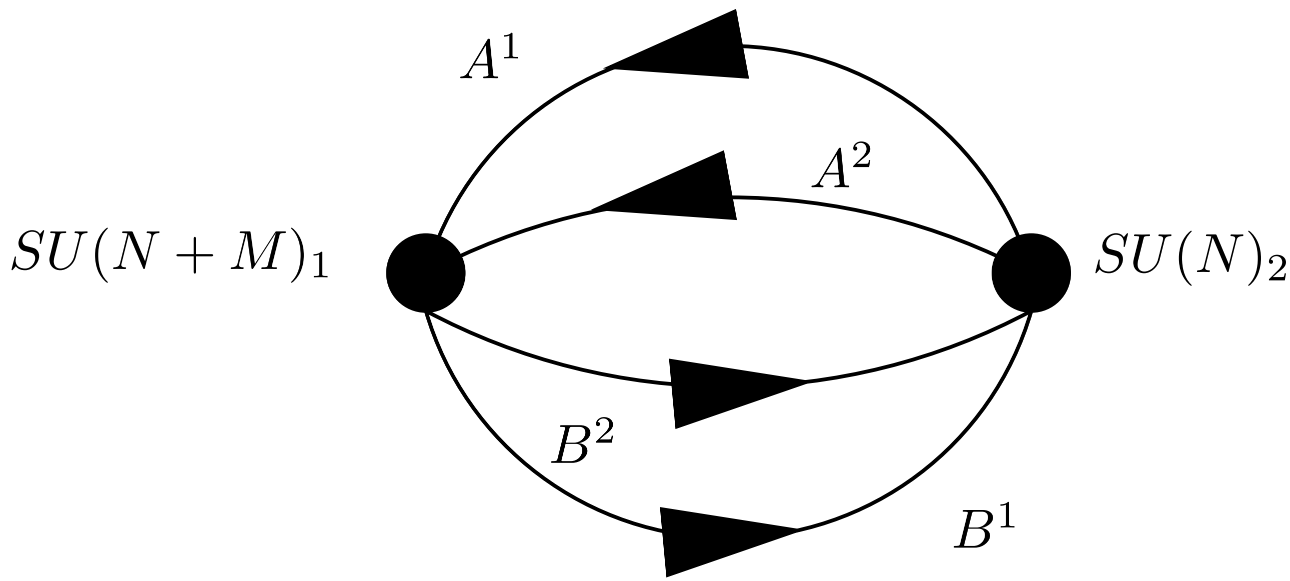
$$[B^i]^{UV} = 1 \quad \rightarrow \quad [B^i]^{IR} = \frac{3}{4} \qquad \Longrightarrow \ \gamma_{B^i} = -\frac{1}{4}$$

$\Longrightarrow$ From the exact NSWZ gauge β-functions (accounting for the anomalous dim of fields) we find

$$\beta_i = 0$$

- Consider a discrete deformation

$$SU(N)_1 \quad \to \quad SU(N+M)_1 \,, \qquad M \ll N$$



$$\beta_1 \sim 3(N+M) - 2N(1-\gamma_{A^i}-\gamma_{B^j}) = 3M + \mathcal{O}(M^3/N^2)$$

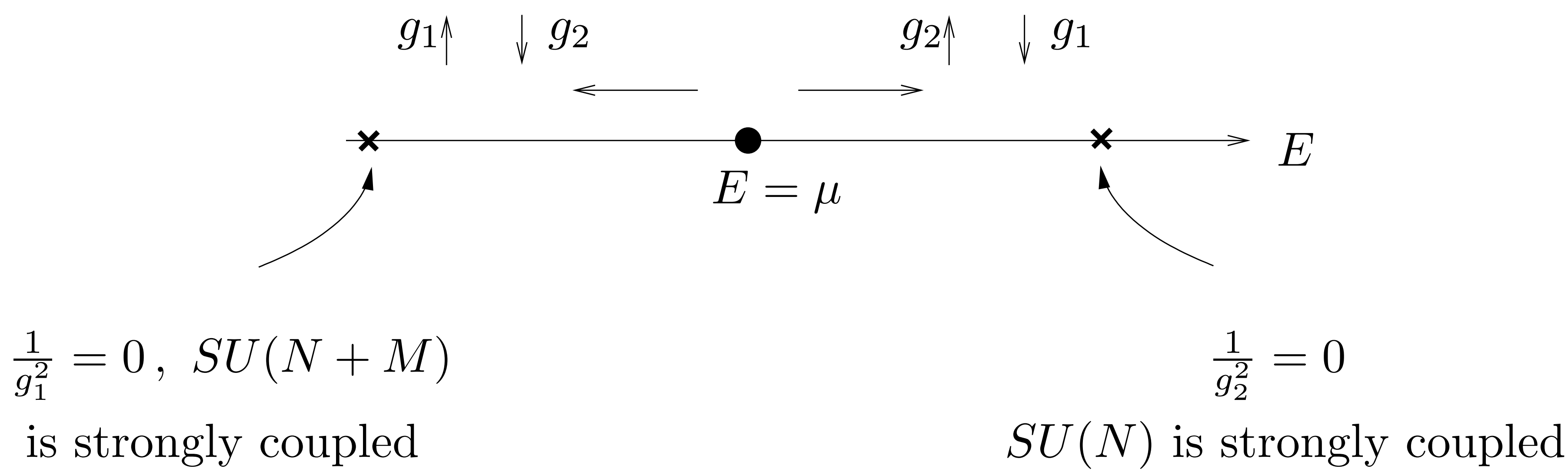
$$\beta_2 \sim 3N - 2(N+M)(1-\gamma_{A^i}-\gamma_{B^j}) = -3M + \mathcal{O}(M^3/N^2)$$

From the β-functions:

$$\frac{4\pi}{g_1^2(\mu)} + \frac{4\pi}{g_2^2(\mu)} = \text{const}$$

$$\frac{4\pi}{g_1^2(\mu)} - \frac{4\pi}{g_2^2(\mu)} \sim M \;\; \ln\frac{\mu}{\Lambda}$$

where Λ is the strong coupling scale of the theory



What is the effective description of the theory past the Landau poles?

$\Longrightarrow$ Using Seiberg duality for $\mathcal{N} = 1$ SUSY gauge theory, the extension of the model past the Landau poles results in self-similarity cascade (Klebanov and Strassler):

$$N \to N(\mu) \sim 2M^2 \;\ln \frac{\mu}{\Lambda}$$

$$\mathrm{UV}: \qquad N \to N + M\,, \qquad\qquad \mathrm{IR}: \qquad N \to N - M$$

$\Longrightarrow$ If N is a multiple of M, the theory in the deep infrared is $\mathcal{N} = 1$ $SU(M)$ SYM; this theory confines with the spontaneous chiral $U(1)_R$ symmetry breaking

Klebanov-Strassler model (a supergravity story)

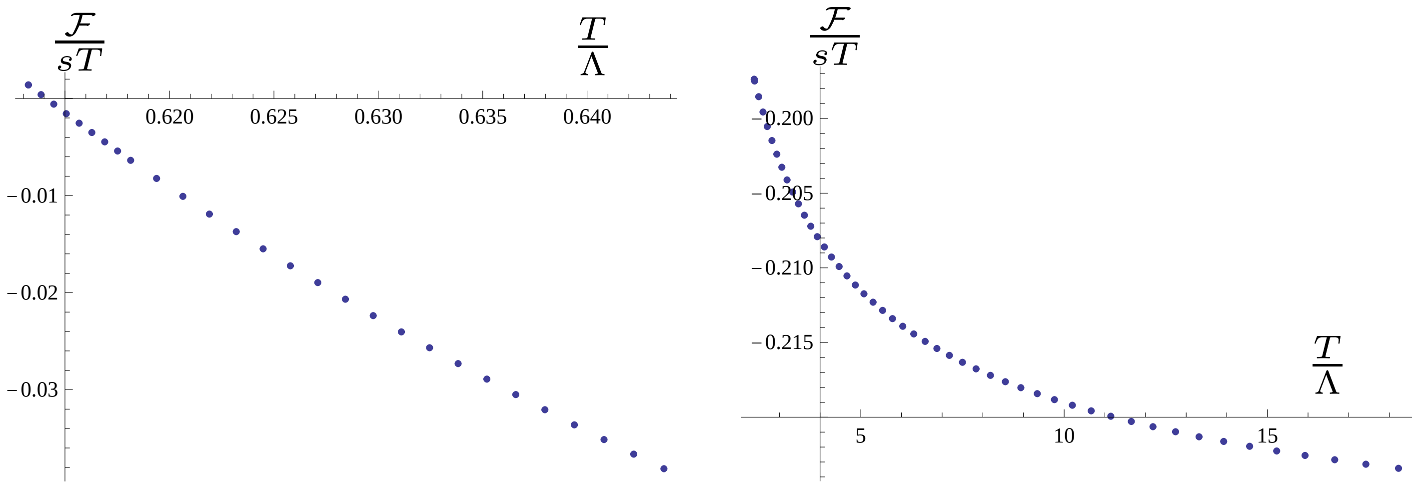
It is possible to derive an effective 5d action from string theory dual to KS model in the deconfined phase with unbroken chiral symmetry:

$$S = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \biggl(R - \frac{40}{3}(\partial f)^2 - 20(\partial w)^2 - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{4M^2}(\partial K)^2 e^{-\Phi-4f-4w} - \mathcal{P} \biggr)$$

$$\mathcal{P} = -24 e^{-\frac{16}{3}f-2w} + 4e^{-\frac{16}{3}f-12w} + M^2 e^{\Phi-\frac{28}{3}f+4w} + \frac{1}{2}K^2 e^{-\frac{40}{3}f}$$

$\Longrightarrow$ The 4 supergravity scalars $\{\Phi, f, w, K\}$ encode operators of dim=$\{4, 4, 6, 8\}$.

$\Longrightarrow$ It is possible (though quite technical) to repeat thermodynamic analysis analogous to those of the $\mathcal{N} = 2^*$ model.



The free energy density $\mathcal{F}$, divided by sT, as a function of $\frac{T}{\Lambda}$. On the left we plot temperatures at and slightly above the deconfinement transition, and on the right much higher temperatures. Note:

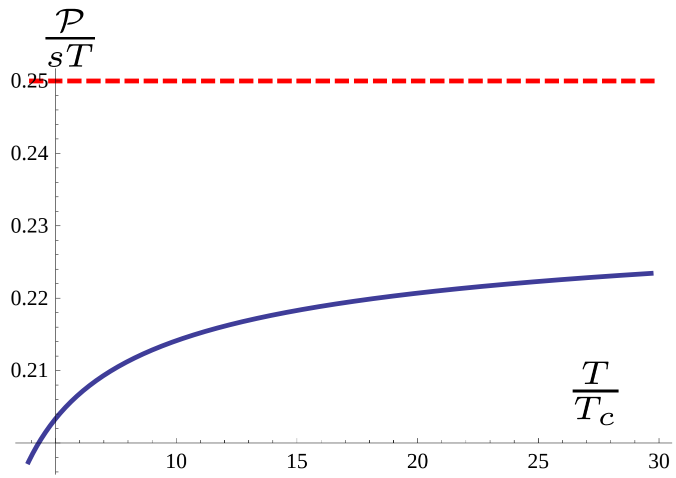
$$\left(\frac{T}{\Lambda}\right)_{deconfinement} = 0.614111(3)$$

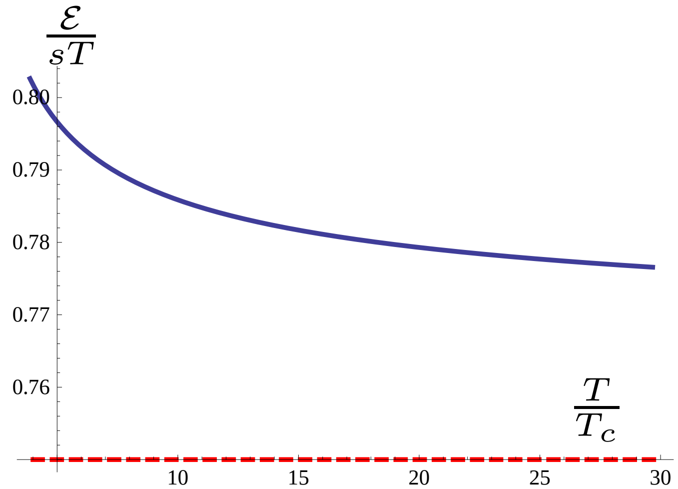
$\Longrightarrow$ The phase transition is of the first-order, between the **deconfined chirally symmetric** phase and the **confined phase with broken chiral symmetry**

Note:

$$\left.\frac{\mathcal{F}}{sT}\right|_{conformal} = -\frac{1}{4}$$

In cascading plasma:





The pressure $\mathcal{P}$ and the energy density $\mathcal{E}$, divided by sT, as a function of $\frac{T}{T_c}$. T_c is the temperature for the deconfinement phase transition in the cascading plasma.

$\Longrightarrow$ No 'thermodynamic plateau' near the deconfinement transition!

$\Longrightarrow$ Is the deconfined chirally symmetric phase of the cascading plasma perturbatively stable?

$\Longrightarrow$ To answer this question:

▪ we look at linearized χsb fluctuations $\propto e^{-i\omega t+i\vec{k}\cdot\vec{x}}$ about chirally symmetric thermal state. Suppose that these fluctuations have a dispersion relation

$$\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)\,, \qquad \mathfrak{w} \equiv \frac{\omega}{2\pi T}\,, \qquad \mathfrak{q} = \frac{|\vec{k}|}{2\pi T}$$

▪ These χsb fluctuations are unstable, provided

$$\mathrm{Im}\,(\mathfrak{w}) > 0 \qquad \text{for} \qquad \mathrm{Im}\,(\mathfrak{q}) = 0$$

▪ Using the holographic duality, one can precisely map these fluctuations into quasinormal modes of the 5d black hole solution, describing the deconfined chirally symmetric equilibrium phase of the cascading plasma

First, we need to augment the holographic effective action to include modes responsible for the chiral symmetry breaking:

$$S_5 = \frac{108}{16\pi G_5}\int_{\mathcal{M}_5} d^5\xi\sqrt{-g}\;\Omega_1\Omega_2^2\Omega_3^2\,\left\{R_{10} - \frac{1}{2}\left(\nabla\Phi\right)^2 - \frac{1}{2}e^{-\Phi}\left(\frac{(h_1-h_3)^2}{2\Omega_1^2\Omega_2^2\Omega_3^2}\right.\right.$$

$$\left. + \frac{1}{\Omega_3^4}\left(\nabla h_1\right)^2 + \frac{1}{\Omega_2^4}\left(\nabla h_3\right)^2\right) - \frac{1}{2}e^{\Phi}\left(\frac{2}{\Omega_2^2\Omega_3^2}\left(\nabla h_2\right)^2 + \frac{1}{\Omega_1^2\Omega_2^4}\left(h_2 - \frac{M}{9}\right)^2\right.$$

$$\left.\left. + \frac{1}{\Omega_1^2\Omega_3^4}h_2^2\right) - \frac{1}{2\Omega_1^2\Omega_2^4\Omega_3^4}\left(4\Omega_0 + h_2\left(h_3 - h_1\right) + \frac{1}{9}Mh_1\right)^2\right\},$$

where:

$$R_{10} = R_5 + \left(\frac{1}{2\Omega_1^2} + \frac{2}{\Omega_2^2} + \frac{2}{\Omega_3^2} - \frac{\Omega_2^2}{4\Omega_1^2\Omega_3^2} - \frac{\Omega_3^2}{4\Omega_1^2\Omega_2^2} - \frac{\Omega_1^2}{\Omega_2^2\Omega_3^2}\right) - 2\Box\ln\left(\Omega_1\Omega_2^2\Omega_3^2\right)$$

$$-\left\{\left(\nabla\ln\Omega_1\right)^2 + 2\left(\nabla\ln\Omega_2\right)^2 + 2\left(\nabla\ln\Omega_3\right)^2 + \left(\nabla\ln\left(\Omega_1\Omega_2^2\Omega_3^2\right)\right)^2\right\},$$

$\Longrightarrow$ There are 3 additional modes: 2 are dual to dimension-3 operators (the gaugino bilinears) plus we are forced to include also a dimension-7 operator that couples them together.

We introduce

$$h_1 = \frac{1}{M}\left(\frac{K_1}{12} - 36\Omega_0\right)\,, \qquad h_2 = \frac{M}{18}\ K_2\,, \qquad h_3 = \frac{1}{M}\left(\frac{K_3}{12} - 36\Omega_0\right)$$

$$\Omega_1 = \frac{1}{3} f_c^{1/2} h^{1/4}\,, \qquad \Omega_2 = \frac{1}{\sqrt{6}} f_a^{1/2} h^{1/4}\,, \qquad \Omega_3 = \frac{1}{\sqrt{6}} f_b^{1/2} h^{1/4}\,.$$

In it possible to verify that linearized fluctuations $\{\delta f, \delta k_1, \delta k_2\}$ in

$$K_1 = K + \delta k_1\,, \qquad K_2 = 1 + \delta k_2\,, \qquad K_3 = K - \delta k_1\,,$$

$$f_c = f_2\,, \qquad f_a = f_3 + \delta f\,, \qquad f_b = f_3 - \delta f\,,$$

decouple from all the other fluctuations, provided the gravitational fields

$$\left\{ds_5^2\,, K\,, h\,, f_2\,, f_3\,, g_s\right\}$$

are on-shell, *i.e.*, describe a chirally symmetric state of the cascading plasma.

The effective action for the χSB fluctuations can be derived from the full effective action:

$$S_{\chi\mathrm{SB}}\left[\delta f, \delta k_1, \delta k_2\right] = \frac{1}{16\pi G_5}\int_{\mathcal{M}_5} \mathrm{vol}_{\mathcal{M}_5}\ h^{5/4} f_2^{1/2} f_3^2 \left\{\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right\}$$

- $$\mathcal{L}_1 = -\frac{(\delta f)^2}{f_3^2}\left(-\frac{P^2 e^{\Phi}}{2 f_2 h^{3/2} f_3^2} - \frac{(\nabla K)^2}{8 f_3^2 h P^2 e^{\Phi}} - \frac{K^2}{2 f_2 h^{5/2} f_3^4}\right)$$

- $$\mathcal{L}_2 = -\frac{9 f_3^2 - 24 f_2 f_3 + 4 f_2^2}{f_2 h^{1/2} f_3^4}\ (\delta f)^2 + 2\Box \frac{(\delta f)^2}{f_3^2} - \left(\nabla \frac{(\delta f)^2}{f_3^2}\right)^2$$
$$-2\nabla\left(\ln h^{1/4} f_3^{1/2}\right)\nabla\left(\frac{(\delta f)^2}{f_3^2}\right) + 2\nabla\left(\ln f_2^{1/2} h^{5/4} f_3^2\right)\nabla\left(\frac{(\delta f)^2}{f_3^2}\right)$$

- $$\mathcal{L}_3 = -\frac{1}{2P^2 e^{\Phi}}\left(\frac{9}{2 f_2 h^{3/2} f_3^2}\ (\delta k_1)^2 + \frac{1}{2 h f_3^4}\left(2(\nabla K)^2\ (\delta f)^2 + f_3^2\ (\nabla \delta k_1)^2 + 4 f_3 \delta f\ \nabla K \nabla \delta k_1\right)\right)$$

- $$\mathcal{L}_4 = \frac{P^2 e^{\Phi}}{2}\left(\frac{2}{9 h f_3^2}\ (\nabla \delta k_2)^2 + \frac{2}{f_2 h^{3/2} f_3^4}\left(3\ (\delta f)^2 + 4 f_3\ \delta f \delta k_2 + f_3^3\ (\delta k_2)^2\right)\right)$$

- $$\mathcal{L}_5 = \frac{K}{f_2 h^{5/2} f_3^6}\ \left(f_3^2\ \delta k_1 \delta k_2 - K\ (\delta f)^2\right)$$

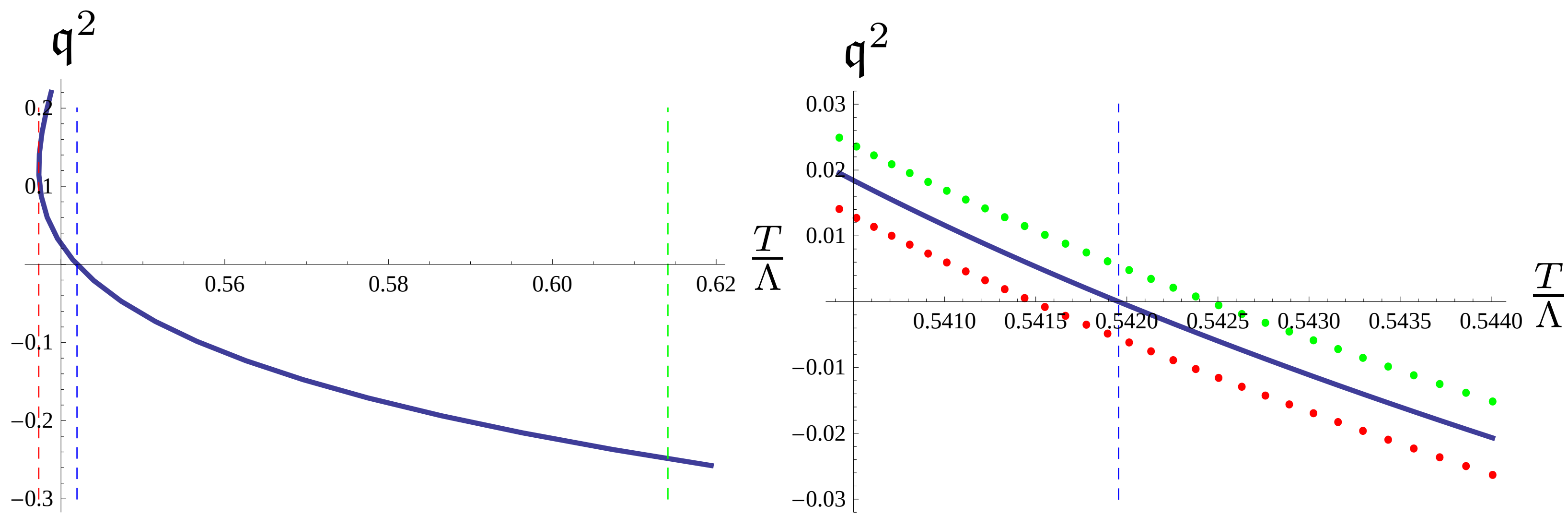
To study the spectrum of fluctuations:

- We derive EOMs from $S_{\chi\mathrm{SB}}$ for $\delta k_i, \delta f$:
- Assume the ansatz

$$\delta f = e^{-i\omega t + ikx_3} F(x)$$

$$\delta k_1 = e^{-i\omega t + ikx_3} \mathcal{K}_1(x)\,, \qquad \delta k_2 = e^{-i\omega t + ikx_3} \mathcal{K}_2(x)$$

- Numerically solve for $\{F, \mathcal{K}_i\}$ on top of numerical KT BH background, for given $\mathfrak{w} = \frac{\omega}{2\pi T}$ and $\frac{T}{\Lambda}$ to determine $\mathfrak{q}^2 = \left(\frac{k}{2\pi T}\right)^2$. One has to impose correct boundary conditions:
  - ▪ at the asymptotic boundary, only normalizable components of the fluctuations are turned on;
  - ▪ near the horizon we have **in-falling** boundary conditions (more on this later)
- Given above, $\mathfrak{q}$ is uniquely determined. As long as $\mathfrak{q}^2 \geq 0$, the fluctuations with chosen $\mathfrak{w}$ are physical.



Dispersion relation of the χSB quasinormal modes of the Klebanov-Tseytlin black hole as a function of $\frac{T}{\Lambda}$. The solid blue lines represent the dispersion relation of the χSB fluctuations at the threshold of instability: $(\mathfrak{w} = 0, \mathfrak{q}^2)$. The blue dashed vertical lines represent the onset of instability: $T = T_{\chi\text{SB}}$, such that $(i\mathfrak{w} = 0, \mathfrak{q}^2 = 0)$. The vertical dashed green and red lines indicate $T = T_c$ and $T = T_u$ correspondingly. The green dots indicate quasinormal modes with $(i\mathfrak{w} = -0.01, \mathfrak{q}^2)$ as a function of $\frac{T}{\Lambda}$. The red dots indicate quasinormal modes with $(i\mathfrak{w} = 0.01, \mathfrak{q}^2)$ as a function of $\frac{T}{\Lambda}$.

$$T_{\chi\text{SB}} = 0.882503(0)T_c$$

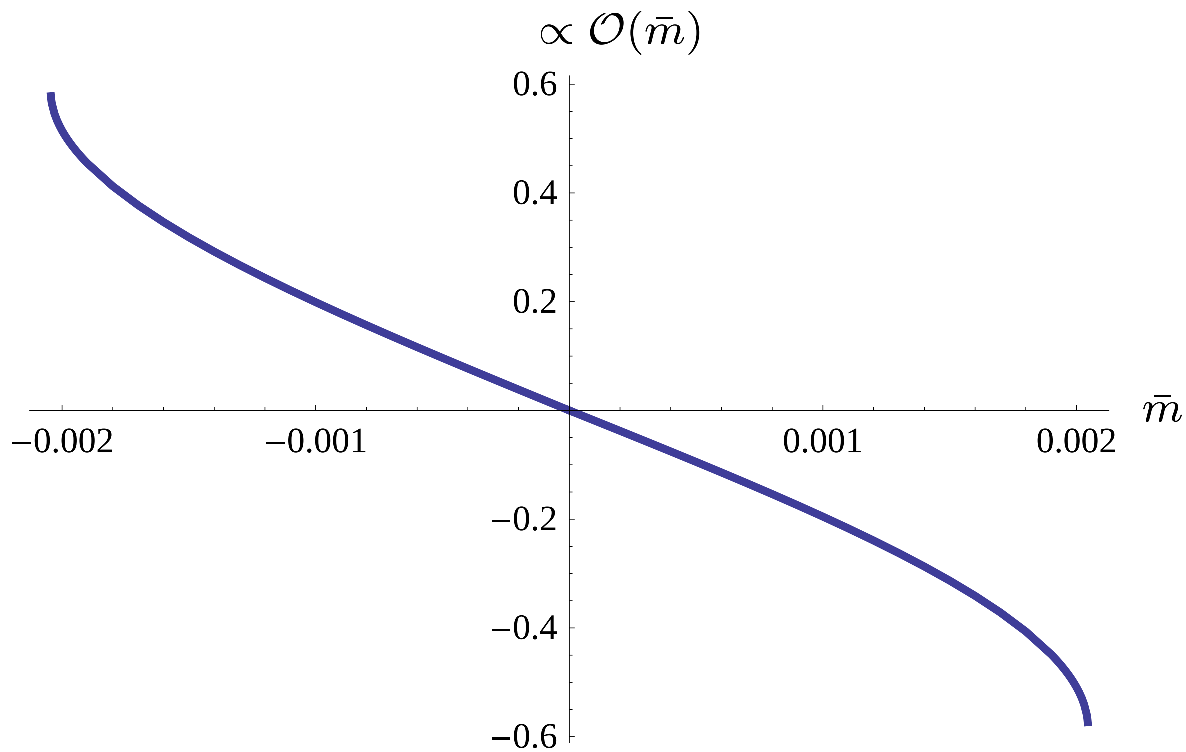
$\Longrightarrow$ If chiral tachyons condense with zero momentum in the new ground state, there must exit homogeneous and isotropic deconfined phase of the cascading plasma, with spontaneous broken chiral symmetry for $T < T_{\chi \mathrm{SB}}$.

$\Longrightarrow$ I will argue now that such homogeneous and isotropic phase does not exist. Specifically, for $T < T_{\chi \mathrm{SB}}$

- we turn the (gaugino) fermion mass $\bar{m} \equiv \frac{M_{fermion}}{T}$, explicitly breaking the chiral symmetry;
- the chiral condensates $T^{-3}\langle\lambda\lambda\rangle \equiv \mathcal{O}(\bar{m})$
- we can compute now

$$\lim_{\bar{m}\to 0} \mathcal{O}(\bar{m})$$

$\Longrightarrow$ Once again, holographic correspondence allows us to compute above expectation value, without any approximation!



$\Longrightarrow$ Homogeneous and isotropic deconfined phase with spontaneously broken chiral symmetry does not exist.

$\Longrightarrow$ It appears chiral tachyons must condense with finite momentum, resulting in some inhomogeneous phase (future work).